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REPRESENTATION VARIETIES OF ARITHMETIC GROUPS AND POLYNOMIAL PERIODICITY OF BETTI NUMBERS

BY

SCOT ADAMS*

Department of Mathematics, University of Chicago
Chicago, Illinois 60673, USA; e-mail: adams@s5.math.umn.edu

ABSTRACT

We give a method of studying character varieties of arithmetic groups with an application to polynomial periodicity of Betti numbers of manifolds in congruence towers.

Introduction

This paper has three main goals. The first is to give a method of studying the character varieties of arithmetic groups. The second is to apply these results to the study of congruence representations, i.e., representations that vanish along some congruence subgroup. The third is to use this study of congruence representations to prove periodicity and polynomial periodicity of Betti numbers of manifolds in a congruence tower, generalizing results of P. Sarnak and Adams [SarAd1]. Another polynomial periodicity result for Betti numbers (but for complex surfaces in branched coverings) has been proved by E. Hironaka [Hir1].

We elaborate on each of these three goals in turn:

1. CHARACTER VARIETIES. Let $\mathcal{H}$ be a linear algebraic $\mathbb{Q}$-group. Let H be a subgroup of $\mathcal{H}(\mathbb{Q})$ which is commensurable with $\mathcal{H}(\mathbb{Z})$. We aim to study the irreducible finite dimensional complex representations of H.

Let $\mathcal{N}$ denote the unipotent radical of $\mathcal{H}$ and let $\mathcal{G}$ be a reductive Levi factor of $\mathcal{H}$ defined over $\mathbb{Q}$. Choose subgroups $G \subseteq \mathcal{G}(\mathbb{Q})$ and $N \subseteq \mathcal{N}(\mathbb{Q})$ which are commensurable with $\mathcal{G}(\mathbb{Z})$ and $\mathcal{N}(\mathbb{Z})$, respectively. By replacing G and N by

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subgroups of finite index, we may assume that $GN \subseteq H$, that the Zariski closure of G is connected and that N is a normal subgroup of H. Note that $G \cap N$ is finite and that the index $[H : GN]$ of GN in H is finite.

First we study the representations of G. Then we fix a representation of G and attempt to understand the possible extensions of that representation from G to an irreducible representation of H. When the representation of G has finite image, much can be said; otherwise the situation is sometimes unclear. (However, the case of finite image is the most important one for our eventual purposes, since every congruence representation has finite image.)

We will comment on congruence representations of G in Part 2 of the introduction. The general case is quite interesting, provided one is willing to pass to finite covers and finite index subgroups. Let $\mathcal{S}$ and $\mathcal{T}$ be the semisimple and toral parts of the connected component $\mathcal{G}^0$ of the identity in $\mathcal{G}$. Let $\mathcal{S}'$ be the product of the $\mathbb{R}$-isotropic, almost $\mathbb{Q}$-simple factors of $\mathcal{S}$. Then we may choose subgroups $S' \subseteq \mathcal{S}'(\mathbb{Q})$ and $T \subseteq \mathcal{T}(\mathbb{Q})$ which are commensurable with $\mathcal{S}'(\mathbb{Z})$ and $\mathcal{T}(\mathbb{Z})$ and satisfy: $S'T \subseteq G$. Then the multiplication map $S' \times T \to G$ has finite index image. By Lemma 3.6 (see also Lemma 3.4 and Corollary 3.5) the irreducible representations of G and of $S' \times T$ are closely related. Now T is a finitely generated Abelian group; its irreducible representations are understood, so we are reduced to studying the representations of S'. If each almost $\mathbb{Q}$-simple factor of $\mathbb{Q}$-rank one in $\mathcal{S}'$ is locally isomorphic either to some Sp_{1n} or to $\mathbf{F}_4^{-20}$, then super-rigidity techniques ([Mar1], [Cor1], [GrScl]) give a very good description of the representations of S': there are the representations that come from "arithmetic" constructions and any representation agrees with one of these along some subgroup of finite index.

Now let V be a finite dimensional complex vector space and let η: $G \to \mathrm{GL}(V)$ be a (possibly reducible) representation of G on V. Let $\mathbb{C}^*$ be the multiplicative group $\mathbb{C}\backslash\{0\}$. Let X be the kernel of the restriction map $\mathrm{Hom}(H, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$. Let R denote the set of all representations τ: $H \to \mathrm{GL}(V)$ which extend η. Let R^s denote the elements of R which are irreducible representations of H. We can define an action of X on R by the formula $(\chi.\tau)(h) = \chi(h)\tau(h)$. Then R^s is X-invariant.

Statements (1) and (2) of Corollary 3.10 assert:

THEOREM A: *Assume that the Zariski closure of G in $\mathcal{G}$ is connected, that $\eta(G)$ is finite and that $H = GN$. Then there exists a finite set $F \subseteq R^s$ such that:*

for every element of $\tau \in R^s$, there is a unique element $\tau' \in F$ such that τ is isomorphic to some element of $X.\tau'$.

This begs the question of whether two elements of $X.\tau'$ can be isomorphic; this question has a satisfactory answer: By [LM1, Lemma 5.8, p. 85], for each $\tau \in R^s$, there is a finite subgroup X_τ of X such that, for all $\chi, \chi' \in X$, we have: $\chi.\tau$ is isomorphic to $\chi'.\tau$ iff $\chi^{-1}\chi' \in X_\tau$. This is reflected in (3) of Corollary 3.10.

Note that, when X is finite, Theorem A tells us that any representation of G with finite image has only finitely many irreducible extensions to H, up to isomorphism. The computation of X can sometimes be complicated because it requires understanding the Abelianizations of H and of G. Since N is nilpotent, its Abelianization is sometimes more easily understood and we can relate X to $\tilde{A} := \mathrm{Hom}(N, \mathbb{C}^*)$ in the following way: Let $\tilde{A}^G$ denote the G-invariant points of $\tilde{A}$. Then every element of X restricts to an element of $\tilde{A}^G$. If $H = GN$, then the restriction map $X \to \tilde{A}^G$ is an isomorphism. So if $\tilde{A}^G$ is finite, then X is finite as well. These remarks are summarized in Corollary 3.11 and improved in Theorem 3.12.

This criterion allows us to conclude, for example: If $k \geq 2$ is an integer, if $\mathcal{G}_a^{k\times 1}$ denotes the additive $\mathbb{Q}$-group of $k \times 1$ column matrices, and if $\mathcal{H} = \mathrm{SL}_k \ltimes \mathcal{G}_a^{k\times 1}$, then X is finite. This tells us that any finite dimensional complex representation of $\mathrm{SL}_k(\mathbb{Z})$ with finite image has only finitely many irreducible extensions to a representation of $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$. G. Glauberman has pointed out to me that elementary techniques prove this specific fact about $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$. Its use here is simply to illustrate Theorem A.

Note: If $k \geq 3$, then there are elementary techniques which show that, if a finite dimensional complex representation of $\mathrm{SL}_k(\mathbb{Z})$ has precompact image, then it has finite image. More generally, super-rigidity and arithmeticity techniques prove that: if Γ is a non-cocompact, irreducible lattice in a semisimple real Lie group which either is of split rank ≥ 2 or is locally isomorphic to $\mathrm{Sp}_{1n}(\mathbb{R})$ or $\mathbf{F}_4^{-20}(\mathbb{R})$, then any finite dimensional complex representation of Γ with precompact image actually has finite image.

We assumed in Theorem A that $\eta(G)$ is finite. This assumption is important, because, by Theorem 2.8, it gives us the cohomology injectivity condition in Theorem 3.9, which is the main ingredient in Corollary 3.10. When $\eta(G)$ is not finite, this injectivity condition may fail, as in Example 2.9. It would be interesting in this example to try to parameterize the extensions of η to irreducible representations of $H = G \ltimes N$; I am not sure whether or not this problem is tractable.

2. Congruence representations. Let $m > 0$ be an integer. Let $\mathcal{H}$ be a $\mathbb{Q}$-subgroup of SL_m. For each integer $K > 0$, let $\mathcal{H}(K)$ denote the elements of $\mathcal{H}(\mathbb{Z})$ which are congruent to the identity matrix modulo K. A finite dimensional complex representation of $\mathcal{H}(\mathbb{Z})$ will be said to be a **congruence representation for** $\mathcal{H}$ if its kernel contains some $\mathcal{H}(K)$. Our goal is to understand all congruence representations for $\mathcal{H}$.

Let $\mathcal{N}$ be the unipotent radical of $\mathcal{H}$ and let $\mathcal{G}$ be a reductive Levi factor of $\mathcal{H}$ defined over $\mathbb{Q}$. As in Part 1 of the introduction, we will first study the congruence representations for $\mathcal{G}$, then fix such a congruence representation $\eta\colon \mathcal{G}(\mathbb{Z}) \to \mathrm{GL}(V)$ for $\mathcal{G}$ and study the extensions of η to irreducible congruence representations for $\mathcal{H}$.

When $\mathcal{G}$ has nontrivial central toral part, the study of its congruence representations is more complicated than what we wish to consider here. For example, if $d > 0$ is an integer not equal to the square of another integer and if

$$\mathcal{G}_0\colon = \left\{ \begin{bmatrix} x & y \\ 2y & x \end{bmatrix} : x^2 - dy^2 = 1 \right\},$$

then elements of $\mathcal{G}_0(\mathbb{Z})$ are the same as integer solutions to Pell's equation $x^2 - dy^2 = 1$. For the purposes of studying Betti numbers, it will be necessary for various reasons to assume that $\mathcal{H}$ has strong approximation; this precludes the possibility that $\mathcal{G}$ contains a nontrivial central torus. In fact, we indicate at the end of Appendix 2 (§8) why the results in this paper will fail if $\mathcal{H} = \mathcal{G}_0$.

Therefore we assume that $\mathcal{G}$ is semisimple, in which case the main result (Theorem 5.12) about $\mathcal{G}$ is:

Theorem B: *If $\mathcal{G}$ is semisimple, then, for any integer $\nu > 0$, there are only finitely many ν-dimensional congruence representations for $\mathcal{G}$.*

This result is due to Z. Rudnick [Appendix to SarAd1], although, for clarity of exposition, his proof immediately specializes to the case $\mathcal{G} = \mathrm{SL}_n$, for some integer $n > 0$. Elaborating on his work, and following some organizing ideas in an unpublished letter from J.-P. Serre to Sarnak, we give here a detailed proof in the general case.

Let $k \geq 2$ be an integer. We observed in Part 1 of the introduction that: Any finite dimensional complex representation of $\mathrm{SL}_k(\mathbb{Z})$ with finite image has only finitely many irreducible extensions to $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$.

Now assume that $k \geq 3$. Then the congruence subgroup property [BMS1] shows that any finite image representation of $\mathrm{SL}_k(\mathbb{Z})$ is a congruence representation and Theorem B says there are only finitely many of these in any dimension. Putting all this together, we can conclude that the group $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$ has only finitely many irreducible finite image representations in any dimension. Finally, since a finite image representation is completely reducible, it follows that $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$ has only finitely many finite image representations in any dimension, irreducible or not. R. Zimmer has remarked to me that this also follows from the fact that $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$ has Kazhdan's property (T). As before, elementary techniques show: any representation of $\mathrm{SL}_k(\mathbb{Z}) \ltimes \mathbb{Z}^{k\times 1}$ with precompact image actually has finite image.

We now apply some of the results of Part 1 of the introduction to understand congruence representations.

Let V be a finite dimensional complex vector space and let $\eta\colon \mathcal{G}(\mathbb{Z}) \to \mathrm{GL}(V)$ be a (possibly reducible) congruence representation for $\mathcal{G}$. Let $\mathbb{T}$ denote the multiplicative group of complex numbers of modulus one. Let X_1 denote the kernel of the restriction map $\mathrm{Hom}(\mathcal{H}(\mathbb{Z}), \mathbb{T}) \to \mathrm{Hom}(\mathcal{G}(\mathbb{Z}), \mathbb{T})$. Let X_c denote the elements of X_1 which are trivial along $\mathcal{H}(K)$, for some integer $K > 0$. Let R_c denote the set of representations $\tau\colon \mathcal{H}(\mathbb{Z}) \to \mathrm{GL}(V)$ which extend η and are trivial along some $\mathcal{H}(K)$. Let R_c^s denote the elements of R_c which are irreducible representations of $\mathcal{H}(\mathbb{Z})$. We can define an action of X_c on R_c by the formula $(\chi.\tau)(h) = \chi(h)\tau(h)$. Then R_c^s is X_c-invariant.

Statements (1) and (2) of Theorem 5.13 assert:

THEOREM C: *Assume that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$ and that the Zariski closure of $\mathcal{G}(\mathbb{Z})$ is connected. Suppose further that $\mathcal{G}$ is semisimple. Then there exists a finite set $F \subseteq R^s$ such that: for every element $\tau \in R_c^s$, there is a unique element $\tau' \in F$ such that τ is isomorphic to some element of $X_c.\tau'$.*

This theorem requires the existence of a $\mathbb{Q}$-Levi factor $\mathcal{G}$ satisfying $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$. Note that there exists a $\mathbb{Q}$-subgroup of SL_5 which does not admit such a Levi factor (see Appendix 1, Example 7.2).

By [LM1, Lemma 5.8, p. 85], for each $\tau \in R_c^s$, there is a finite subgroup X_τ of

X_c such that, for all $\chi, \chi' \in X_c$, we have: $\chi.\tau$ is isomorphic to $\chi'.\tau$ iff $\chi^{-1}\chi' \in X_\tau$. This yields (3) of Theorem 5.13.

When X_c is finite, Theorem C implies that any congruence representation for $\mathcal{G}$ has only finitely many extensions to an congruence representation for $\mathcal{H}$, up to isomorphism. Any congruence representation has finite image and is therefore completely reducible. Combining this with Theorem B, we have:

THEOREM D: *Assume that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$ and that the Zariski closure of $\mathcal{G}(\mathbb{Z})$ is connected. Suppose further that $\mathcal{G}$ is semisimple and that X_c is finite. Then $\mathcal{H}$ has only finitely many congruence representations in any given dimension.*

Even without the assumption that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$, there is still a version (Corollary 5.14) of Theorem D that is valid.

There is one more result about congruence representations that is important to the periodicity and polynomial periodicity results we wish to obtain for Betti numbers in a congruence tower. This is given by Corollary 5.4 and Theorem 5.6 which, together, imply:

THEOREM E: *Assume that $\mathcal{H}$ is connected and algebraically simply connected. Assume that every almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic. Then, for all congruence representations τ for $\mathcal{H}$, there exists an integer $K_\tau > 0$ such that, for all integers $K > 0$, we have: $\mathcal{H}(K) \subseteq \ker(\tau)$ iff $K_\tau | K$. Furthermore, for any congruence representation τ for $\mathcal{H}$, there exists $\chi \in X_c$ such that, for all $\chi' \in X_c$, we have: $K_{\chi.\tau} | K_{\chi'.\tau}$.*

This result yields (4) of Theorem 5.13.

In Theorem E, the assumption that $\mathcal{H}$ is connected and algebraically simply connected is equivalent to requiring that the complex points of $\mathcal{H}$ form a connected, simply connected complex Lie group. It is important to insure that $\mathcal{H}$ has the strong approximation property.

The assumption that every almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic is equivalent to requiring that $\mathcal{G}(\mathbb{Z})$ has dense projection in any compact quotient of the semisimple real Lie group $\mathcal{G}(\mathbb{R})$ (see [Joh1, Theorem 3.4, p. 64]). It is important to guarantee that the strong approximation property holds at the infinite place.

3. BETTI NUMBERS. Let $m > 0$ be an integer. Let $\mathcal{H}$ be an algebraic $\mathbb{Q}$-subgroup of SL_m. Let $\mathcal{N}$ denote the unipotent radical of $\mathcal{H}$. Assume that the

strong approximation theorem holds for $\mathcal{H}$ at the infinite place. That is, we again require that $\mathcal{H}/\mathcal{N}$ be a connected, algebraically simply connected, semisimple $\mathbb{Q}$-group all of whose almost $\mathbb{Q}$-simple factors are $\mathbb{R}$-isotropic. Let $H\colon = \mathcal{H}(\mathbb{Z})$. For each integer $K > 0$, let $\mathcal{H}(K)$ denote matrices in H which are congruent to the identity matrix modulo K and let $H_K\colon = \mathcal{H}(\mathbb{Z})/\mathcal{H}(K)$.

Let M be a connected topological space with the homotopy type of a finite CW complex. Let $\tilde{M}$ denote the universal cover of M and let $\Gamma := \pi_1(M)$ denote the fundamental group of M. Fix a surjective homomorphism $F\colon \Gamma \to H$.

Let $K, q > 0$ be integers. Let $M_K := \tilde{M} \times_\Gamma H_K$ denote the covering of M associated to H_K and let $F_K\colon \Gamma \to H_K$ denote the composite of F and the natural map $H \to H_K$. For any finite dimensional representation τ of H_K, let $\beta^q_\Gamma(\tilde{M}, \tau)$ denote the qth twisted Betti number of M with a local coefficient system coming from the flat bundle defined by $\tau \circ F_K$. By Corollary 6.2, we have the formula: $\beta^q(M_K) = \sum \dim(\tau) \cdot \beta^q_\Gamma(\tilde{M}, \tau)$, where the sum is taken over the irreducible representations τ of the finite group H_K. For any integer $\nu > 0$, we define $\beta^{q,\nu}_K := \nu \cdot \sum \beta^q_\Gamma(\tilde{M}, \tau)$, where the sum is now taken only over those irreducible representations τ of H_K such that $\dim(\tau) = \nu$.

In Theorem 6.7, we give our application. It states:

THEOREM F: *Fix integers $q, \nu > 0$. Let $\mathcal{N}$ denote the unipotent radical of $\mathcal{H}$. Assume that $\mathcal{H}$ has a $\mathbb{Q}$-Levi factor $\mathcal{G}$ such that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$. Then the sequence $\beta^{q,\nu}_1, \beta^{q,\nu}_2, \ldots$ is* **polynomial periodic**, *i.e., there exists an integer $s > 0$ and there exists a finite sequence $Q_1, \ldots, Q_s\colon \mathbb{R} \to \mathbb{R}$ of polynomial functions such that: if a, r are integers, if $a \geq 0$ and if $1 \leq r \leq s$, then $\beta^{q,\nu}_{as+r} = Q_r(a)$.*

Let $\mathbb{T}$ denote the multiplicative group of complex numbers of modulus one. We also have a periodicity result (see Theorem 6.8) which is based on Theorem D and on the proof of Theorem F:

THEOREM G: *Let $\mathcal{G}$ denote a $\mathbb{Q}$-Levi factor of $\mathcal{H}$. Let $N := \mathcal{N}(\mathbb{Z})$ and $G := \mathcal{G}(\mathbb{Z})$. Then G acts on N by conjugation, inducing an action of G on $A := N/[N, N]$. This, in turn, induces an action of G on $\hat{A} := \mathrm{Hom}(A, \mathbb{T})$. Assume that there are only finitely many G-fixpoints in $\hat{A}$. Then, for all integers $q, \nu > 0$, the sequence $\beta^{q,\nu}_1, \beta^{q,\nu}_2, \ldots$ is periodic.*

For example, if $\mathcal{H}$ is semisimple (i.e., if $\mathcal{N}$ is trivial), then the preceding periodicity result holds. For another example: If $k \geq 2$ is any integer, if $\mathcal{G}^{k\times 1}_a$

denotes the additive group of $k \times 1$ column matrices and if $\mathcal{H} = \mathrm{SL}_k \ltimes \mathcal{G}_a^{k\times 1}$, then periodicity holds.

For any numerical invariant I of closed manifolds, if there is a notion of "twisted-I" that depends on a flat bundle over a closed manifold, then there is a good chance that, along the congruence tower $M_1, M_2, \ldots$, there is a decomposition of each $I(M_K)$ into its "dimensional components": $I(M_K) = \sum_{\nu=1}^{\infty} I_\nu(M_K)$ and that, for each integer $\nu > 0$, the sequence $I_\nu(M_1), I_\nu(M_2), \ldots$ is polynomial periodic (assuming the existence of a $\mathbb{Q}$-Levi factor $\mathcal{G}$ such that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$) or even periodic (assuming that $\hat{A}$ has finitely many G fixpoints, as in Theorem G). To obtain these results, it suffices to verify the two conditions on I described below.

Let $M' \to M$ denote a finite Galois covering of M with Galois group F. Let $\hat{F}$ denote the set of irreducible representations of F. For each $R \in \hat{F}$, let $\tilde{R}$ denote the pullback of R to Γ. The **first condition** is that, for any such finite Galois cover M', we have $I(M') = \sum I_\Gamma(\tilde{M}, \tilde{R})$, where the sum is over elements $R \in \hat{F}$.

Let $X_1 := \mathrm{Hom}(\Gamma, \mathbb{T})$. Then X_1 is the dual group of the Abelianization of Γ. The connected component of the identity X_1^0 of X_1 is a compact, connected torus. Let R denote a representation of Γ such that $R(\Gamma)$ is finite. The **second condition** is that, for all such finite image representations R, the map $\chi \mapsto I_\Gamma(\tilde{M}, \chi R)$: $X_1^0 \to \mathbb{R}$ is **Zariski upper semicontinuous**, i.e., each of its super-level sets is the set of common zeroes of some collection of trigonometric polynomials on the torus X_1^0.

T. Mrowka has verified, for all integers $k > 0$, that these two conditions hold when I denotes intersection signatures of closed $2k$-manifolds. We may therefore conclude that, along a congruence tower (as described above), for any integer $\nu > 0$, the ν-dimensional intersection numbers follow a polynomial periodic progression.

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1. Global notation

Throughout this paper, let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and let $\mathcal{P}$ denote the set of all prime numbers. Let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers and let $\mathbb{T} \subseteq \mathbb{C}^*$ denote the subgroup consisting of complex numbers of modulus one.

The center of a group G is denoted $Z(G)$. When $\mathcal{G}$ is a linear algebraic group, its center will be denoted $\mathcal{Z}(\mathcal{G})$.

We will say that a linear algebraic group is **connected** if it is irreducible as a variety.

By **representation**, we will always mean a continuous linear representation on a finite dimensional complex vector space.

If a group G acts on a set S, then the set of points of S that are fixed by every element of G is denoted S^G. If a group G acts on two groups A and B, then $\mathrm{Hom}_G(A, B)$ denotes the set of G-equivariant homomorphisms from A to B. If A is a subgroup of a group B, then $[B : A]$ denotes the index of A in B.

If V is a complex vector space, then $\mathrm{gl}\,V$ denotes the $\mathbb{C}$-algebra of all $\mathbb{C}$-endomorphisms of V and $\mathrm{sl}\,V$ denotes the $\mathbb{C}$-subspace of all elements of $\mathrm{gl}\,V$ of trace zero. Similarly, $\mathrm{GL}(V)$ denotes the complex Lie group of all $\mathbb{C}$-automorphisms of V and $\mathrm{SL}(V)$ denotes the elements of $\mathrm{GL}(V)$ of determinant one.

The **Abelianization** of a topological group A is the quotient $A/\overline{[A, A]}$ of A by the closure of its commutator subgroup.

2. Cohomology theorems

Let $\mathcal{H}$ denote a linear algebraic $\mathbb{Q}$-group. Let $\mathcal{G}$ denote a reductive $\mathbb{Q}$-Levi factor of $\mathcal{H}$. Let $\mathcal{N}$ denote the unipotent radical of $\mathcal{H}$.

Let H, G and N denote subgroups of $\mathcal{H}(\mathbb{Q})$, $\mathcal{G}(\mathbb{Q})$ and of $\mathcal{N}(\mathbb{Q})$ such that

(1) H, G and N are commensurable with $\mathcal{H}(\mathbb{Z})$, $\mathcal{G}(\mathbb{Z})$ and $\mathcal{N}(\mathbb{Z})$, respectively;
(2) the Zariski closure of G in $\mathcal{G}$ is connected;
(3) N is normalized by G; and
(4) $H = GN$.

Our goal in this section is to use these hypotheses to prove a cohomology injectivity result (Theorem 2.8) which is needed in Corollary 3.10. The necessity of these hypotheses is explained in Example 2.10.

For any $\mathbb{C}[H]$-module V, let G act on $H^1(N, V)$ by $(g.f)(n) = g(f(g^{-1}ng))$.

LEMMA 2.1: *Let $\mathcal{W}$ be a $\mathbb{Q}$-vector group. Let W' denote a subgroup of $\mathcal{W}(\mathbb{Q})$. Assume that W' is a discrete subgroup of $\mathcal{W}(\mathbb{R})$. Let $\mathcal{W}'$ denote the Zariski closure of W' in $\mathcal{W}$. Then W' is commensurable with $\mathcal{W}'(\mathbb{Z})$.*

Proof: Both W' and $\mathcal{W}'(\mathbb{Z})$ are contained in $\mathcal{W}'(\mathbb{Q})$ and both W' and $\mathcal{W}'(\mathbb{Z})$ are lattices in $\mathcal{W}'(\mathbb{R})$. ∎

LEMMA 2.2: *Let $\mathcal{G}_0$ denote the Zariski closure of G in $\mathcal{G}$. Let $\mathcal{G}_0$ act $\mathbb{Q}$-algebraically on a $\mathbb{Q}$-variety $\mathcal{Y}$. Let $Y \subseteq \mathcal{Y}(\mathbb{Q})$ be G-invariant. Assume that every G-orbit in Y is finite. Then the action of G on Y is trivial.*

Proof: By assumption (2) above, $\mathcal{G}_0$ is connected. Choose $y \in Y$ and assume that the orbit $G.y$ is finite. We wish to show that the action of G on $G.y$ is trivial. Since G is Zariski dense in $\mathcal{G}_0$ and since $G.y$ is Zariski closed in $\mathcal{Y}$, it follows that $G.y$ is $\mathcal{G}_0$-invariant. But $\mathcal{G}_0$ is connected and $G.y$ is finite, so the action of $\mathcal{G}_0$ on $G.y$ must be trivial. In particular, the action of G on $G.y$ is trivial. ∎

LEMMA 2.3: *Let V be a finite dimensional $\mathbb{C}[G]$-module with finite G-orbits. If $V^G = 0$, then $\operatorname{Hom}_G(Z(N), V) = 0$.*

Proof: Let $f \in \operatorname{Hom}_G(Z(N), V)$. We wish to show that $f \equiv 0$.

Since $Z(N) = N \cap \mathcal{Z}(\mathcal{N})$, we may replace $\mathcal{N}$ by $\mathcal{Z}(\mathcal{N})$ and N by $Z(N)$ and assume that N is Abelian and that $\mathcal{N}$ is a $\mathbb{Q}$-vector group.

Let $\mathcal{G}_0$ be the Zariski closure in $\mathcal{G}$ of G. Let $\mathcal{W}'$ be the Zariski closure in $\mathcal{N}$ of $\ker f$. Since f is G-equivariant, it follows that $\ker f$ is normalized by G and, therefore, that $\mathcal{W}'$ is normalized by $\mathcal{G}_0$.

By Lemma 2.1, $\ker f$ is commensurable with $\mathcal{W}'(\mathbb{Z})$. Since N is commensurable with $\mathcal{N}(\mathbb{Z})$, it follows that $N \cap \mathcal{W}'(\mathbb{Q})$ is commensurable with $\mathcal{N}(\mathbb{Z}) \cap \mathcal{W}'(\mathbb{Q}) = \mathcal{W}'(\mathbb{Z})$, so we find that $\ker f$ is commensurable with $N \cap \mathcal{W}'(\mathbb{Q})$. But $\ker f \subseteq N \cap \mathcal{W}'(\mathbb{Q})$, so $\ker f$ has finite index in $N \cap \mathcal{W}'(\mathbb{Q})$. The map f defines an injection $(N \cap \mathcal{W}'(\mathbb{Q}))/\ker f \hookrightarrow (V, +)$. As $(V, +)$ has no nontrivial finite subgroups, it follows that $N \cap \mathcal{W}'(\mathbb{Q}) = \ker f$.

Since $V^G = 0$, it suffices to show that $f(N) \subseteq V^G$. This is equivalent to showing that the G action on $N/\ker f$ is trivial. Let $\mathcal{Y} := \mathcal{N}/\mathcal{W}'$ and let Y be the image in $\mathcal{Y}(\mathbb{Q})$ of N. Then there is a G-equivariant bijection between Y and $N/\ker f$.

Since all G-orbits in V are finite and since f induces an injective G-map $N/\ker f \to V$, it follows that all G-orbits in $N/\ker f$ are finite. By Lemma 2.2, we are done. ∎

Example 2.4: It is not sufficient to assume in Lemma 2.3 that all G-orbits in V are precompact.

Proof: Let $F(x,y,z) = x^2 + y^2 - \sqrt{2}z^2$ and let $\sigma\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ be the field automorphism such that $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $F^\sigma(x,y,z) = x^2 + y^2 + \sqrt{2}z^2$. For each integer $k > 0$, let $\mathcal{G}_a^{k\times 1}$ denote the additive group of $k \times 1$ column matrices. Let SO_F and SO_{F^σ} denote the algebraic groups of 3×3 matrices with determinant one which preserve F and F^σ, respectively. Let $\mathcal{G} := \mathrm{SO}_F \times \mathrm{SO}_{F^\sigma}$. Let $\mathcal{N} := \mathcal{G}_a^{3\times 1} \times \mathcal{G}_a^{3\times 1}$. Let SO_F and SO_{F^σ} act on $\mathcal{G}_a^{3\times 1}$ by matrix multiplication. Let $\mathcal{G}$ act on $\mathcal{N}$ as the product of these two actions.

By restriction of scalars, there are an integer $n > 0$ and embeddings $\mathcal{G} \subseteq \mathrm{SL}_n$, $\mathcal{N} \subseteq \mathcal{G}_a^{n\times 1}$ both with image defined over $\mathbb{Q}$ such that: using these embeddings to define $\mathcal{G}(\mathbb{Z})$ and $\mathcal{N}(\mathbb{Z})$, we have

(1) the restriction to $\mathcal{G}$ of the action of SL_n on $\mathcal{G}_a^{n\times 1}$ (by matrix multiplication) leaves $\mathcal{N}$ invariant and the resulting action of $\mathcal{G}$ on $\mathcal{N}$ is the one specified above;

(2) the image of

$$M \mapsto (M, M^\sigma)\colon \mathrm{SO}_F(\mathbb{Z}[\sqrt{2}]) \to \mathrm{SO}_F(\mathbb{Z}[\sqrt{2}]) \times \mathrm{SO}_{F^\sigma}(\mathbb{Z}[\sqrt{2}])$$

is $\mathcal{G}(\mathbb{Z})$; and

(3) the image of

$$M \mapsto (M, M^\sigma)\colon \mathcal{G}_a^{3\times 1}(\mathbb{Z}[\sqrt{2}]) \to \mathcal{G}_a^{3\times 1}(\mathbb{Z}[\sqrt{2}]) \times \mathcal{G}_a^{3\times 1}(\mathbb{Z}[\sqrt{2}])$$

is $\mathcal{N}(\mathbb{Z})$.

Let $G := \mathcal{G}(\mathbb{Z})$, $N := \mathcal{N}(\mathbb{Z})$, $V := \mathbb{C}^{3\times 1}$. Let $\eta : G \to \mathrm{GL}(V)$ be the restriction to G of second coordinate projection: $\mathrm{SO}_F(\mathbb{C}) \times \mathrm{SO}_{F^\sigma}(\mathbb{C}) \to \mathrm{SO}_{F^\sigma}(\mathbb{C}) \subseteq \mathrm{GL}_3(\mathbb{C}) = \mathrm{GL}(V)$. The resulting action of G on V has precompact G-orbits and satisfies $V^G = 0$.

Let $f\colon N \to V$ denote the restriction to N of the second coordinate projection morphism: $\mathcal{G}_a^{3\times 1}(\mathbb{C}) \times \mathcal{G}_a^{3\times 1}(\mathbb{C}) \to \mathcal{G}_a^{3\times 1}(\mathbb{C})$. Then $0 \neq f \in \mathrm{Hom}_G(N, V)$. Since $N = Z(N)$, this demonstrates that $\mathrm{Hom}_G(Z(N), V) \neq 0$. ∎

LEMMA 2.5: *Let V be a finite dimensional, irreducible, nontrivial $\mathbb{C}[H]$-module with finite G-orbits. Then $H^1(N,V)^G = 0$.*

Proof: The proof is by induction on $\dim \mathcal{N}$. We will make repeated use of the exactness of the sequence:

$$(*) \qquad 0 \to H^1(N/Z(N), V^{Z(N)}) \to H^1(N,V) \to H^1(Z(N), V)^{N/Z(N)}.$$

As $Z(N)$ is a normal subgroup of H, it follows that $V^{Z(N)}$ is H-invariant, so, by irreducibility, either $V^{Z(N)} = 0$ or $V^{Z(N)} = V$.

The case $V^{Z(N)} = 0$ is easy: the second term of $(*)$ vanishes and, by [Rud1, Example 1.4, p. 263], the fourth term of $(*)$ vanishes; consequently, by exactness, $H^1(N, V) = 0$.

So we may assume that $V^{Z(N)} = V$, i.e., that $Z(N)$ acts trivially on V.

Let $\mathcal{N}' := \mathcal{N}/\mathcal{Z}(\mathcal{N})$ and let $\pi := \mathcal{N} \to \mathcal{N}'$ be the natural $\mathbb{Q}$-morphism. Let $N' := \pi(N)$. Then N' is commensurable with $\mathcal{N}'(\mathbb{Z})$, so, by induction, $H^1(N', V)^G = 0$. On the other hand, there is a G-equivariant isomorphism between $N/Z(N)$ and N'. Since $V = V^{Z(N)}$, we have $H^1(N/Z(N), V^{Z(N)})^G = 0$. By exactness of $(*)$, it now suffices to show that $[H^1(Z(N), V)^{N/Z(N)}]^G = 0$.

Since $Z(N)$ acts trivially on V, we have

$$H^1(Z(N), V)^{N/Z(N)} = \mathrm{Hom}_N(Z(N), V).$$

Further, since N centralizes $Z(N)$, we have $\mathrm{Hom}_N(Z(N), V) = \mathrm{Hom}(Z(N), V^N)$. It therefore suffices to show that $\mathrm{Hom}_G(Z(N), V^N) = 0$.

As N is a normal subgroup of H, we conclude that V^N is H-invariant. As the action of H on V is irreducible, it follows that either $V^N = 0$ or $V^N = V$. If $V^N = 0$, then we are done, so we assume that $V^N = V$, i.e., that the N-action on V is trivial. We now wish to show that $\mathrm{Hom}_G(Z(N), V) = 0$.

Since H acts nontrivially and irreducibly on V and since N acts trivially on V, it follows that $V^G = 0$. The conclusion now follows from Lemma 2.3. ∎

LEMMA 2.6: *Let A and B be subgroups of a countable discrete group and assume that A normalizes B. Let V be a finite dimensional $\mathbb{C}[AB]$-module. Let K_Z, K_B, K_H denote the kernels of the restriction maps*

$$Z^1(B, V) \to Z^1(A \cap B, V), \quad B^1(B, V) \to B^1(A \cap B, V),$$
$$H^1(B, V) \to H^1(A \cap B, V).$$

Let A act on $Z^1(B, V)$, $B^1(B, V)$ and $H^1(B, V)$ by $(a.f)(n) = a.(f(a^{-1}na))$. Then K_Z, K_B and K_H are A-invariant and

(i) *The kernel of $Z^1(AB, V) \to Z^1(A, V)$ is isomorphic to the A-invariants K_Z^A in K_Z;*

(ii) *The kernel of $B^1(AB, V) \to B^1(A, V)$ is isomorphic to the A-invariants K_B^A in K_B;*

(iii) *The kernel of* $H^1(AB,V) \to H^1(A,V)$ *is isomorphic to* K_Z^A/K_B^A*, which is isomorphic to a submodule of* K_H^A.

Proof: Conclusions (i) and (ii) follow from the definitions of Z^1 and B^1.

By definition of B^1, the map $B^1(AB,V) \to B^1(A,V)$ is surjective. Conclusion (iii) therefore follows by appying the Snake Lemma to the restriction map which goes from the short exact sequence

$$0 \to B^1(AB,V) \to Z^1(AB,V) \to H^1(AB,V) \to 0$$

to the short exact sequence

$$0 \to B^1(A,V) \to Z^1(A,V) \to H^1(A,V) \to 0. \qquad \blacksquare$$

COROLLARY 2.7: *Let* V *be a finite dimensional, irreducible, nontrivial* $\mathbb{C}[H]$*-module with finite* G*-orbits. Then the restriction map* $H^1(H,V) \to H^1(G,V)$ *is injective.*

Proof: Let $A := G$ and $B := N$ and define K_H as in Lemma 2.6. Since $G \cap N$ is finite, it follows that $H^1(G \cap N, V) = 0$, so $K_H = H^1(N,V)$. By Lemma 2.5, $K_H^A = 0$, so (iii) of Lemma 2.6 gives the result. $\blacksquare$

THEOREM 2.8: *Let* V *be a finite dimensional* $\mathbb{C}[H]$*-module with finite* G*-orbits. If* $V^H = 0$*, then the restriction map* $H^1(H,V) \to H^1(G,V)$ *is injective.*

Proof: The proof is by induction on $\dim V$.

If V is an irreducible H-module, then we are done by Corollary 2.7. We may therefore assume that V is not irreducible and choose a nonzero, proper, H-invariant subspace V' of V. Let $V'' := V/V'$.

Let Q_G denote the cokernel of the map $V^G \to (V'')^G$ and let Q_H denote the cokernel of the map $V^H \to (V'')^H$. We obtain long exact sequences by applying the (left-exact) functors $W \mapsto W^G$ and $W \mapsto W^H$ to the short exact sequence $0 \to V' \to V \to V'' \to 0$. These long exact sequences give rise to injections $Q_G \hookrightarrow H^1(G,V')$ and $Q_H \hookrightarrow H^1(H,V')$.

Note that $(V')^H \subseteq V^H = 0$. Thus, by induction, we have that the restriction map $H^1(H,V') \to H^1(G,V')$ is injective. Since we have identified Q_G and Q_H as submodules of $H^1(G,V')$ and $H^1(H,V')$, it follows that there exists an injective map $Q_H \to Q_G$.

Since the G-action on V has finite orbits, it follows that the image of G in $\mathrm{GL}(V)$ is precompact. Consequently, there exists a G-invariant Hermitian inner product on V and so there is a G-invariant complement to V' in V. So the map $V \to V''$ has a G-equivariant right inverse and we conclude that every G-invariant element of V'' has a G-invariant preimage in V. That is, $Q_G = 0$.

Since there is an injective map $Q_H \to Q_G$, we find that $Q_H = 0$. Then $V^H \to (V'')^H$ is surjective. By assumption, $V^H = 0$, so $(V'')^H = 0$.

Let K', K and K'' denote the kernels of the restriction maps

$$H^1(H,V') \to H^1(G,V'), \quad H^1(H,V) \to H^1(G,V),$$
$$\text{and} \quad H^1(H,V'') \to H^1(G,V'').$$

Then there is an exact sequence $K' \to K \to K''$. Since $(V')^H = (V'')^H = 0$, we may apply induction to conclude that $K' = K'' = 0$. Then $K = 0$, proving that the restriction map $H^1(H,V) \to H^1(G,V)$ is injective. ∎

Example 2.9: The condition that V has finite G-orbits is necessary in Theorem 2.8.

Proof: Let $\nu > 0$ be an integer, let $G := \mathrm{SL}_\nu(\mathbb{Z})$ and let N denote the additive group $\mathrm{sl}_\nu(\mathbb{Z})$ of traceless $\nu \times \nu$ matrices with entries in $\mathbb{Z}$. Let G act on N by $g.n = gng^{-1}$. Let $H := G \ltimes N$. Let η: $G \to \mathrm{SL}_\nu(\mathbb{C})$ denote the identity map. Let H act on $V := \mathbb{C}^{\nu \times 1}$ by $(gn).v = gv$, for all $g \in G$, $n \in N$ and $v \in V$. Let H act on $\mathrm{sl}\,V$ by $(h.\phi)(v) = h.(\phi(h^{-1}.v))$. We will show that the restriction map $H^1(H,\mathrm{sl}\,V) \to H^1(G,\mathrm{sl}\,V)$ is not injective.

Let $A := G$ and $B := N$ and define K_Z, K_B and K_H as in Lemma 2.6, but with V replaced by $\mathrm{sl}\,V$. Then $K_Z = Z^1(N,\mathrm{sl}\,V)$, $K_B = B^1(N,\mathrm{sl}\,V)$ and $K_H = H^1(N,\mathrm{sl}\,V)$.

The N-action on V is trivial, so the N-action on $\mathrm{sl}\,V$ is trivial, so $K_Z = \mathrm{Hom}(N,\mathrm{sl}\,V)$ and $K_B = 0$. The identity map is a G-equivariant homomorphism from N to $\mathrm{sl}\,V$, so $K_Z^A \neq 0$.

By (iii) of Lemma 2.6, the kernel of $H^1(H,\mathrm{sl}\,V) \to H^1(G,\mathrm{sl}\,V)$ is isomorphic to K_Z^A/K_B^A. But $K_Z^A \neq 0$ while $K_B^A = 0$, so we are done. ∎

Example 2,10: The assumptions (made throughout §2) that G have connected Zariski closure and that $H = GN$ are both necessary in Theorem 2.8.

Proof: Define

$$\mathcal{H} := \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, ab = 1 \right\},$$

$$g_0 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad n_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathcal{G}$ denote the intersection of $\mathcal{H}$ with the group of diagonal matrices of determinant one. Then $G_0 := \mathcal{G}(\mathbb{Z})$ is the two-element group generated by g_0. Let $\mathcal{N}$ denote the intersection of $\mathcal{H}$ with the upper triangular unipotent matrices. Then $N_0 := \mathcal{N}(\mathbb{Z})$ is the infinite cyclic group generated by n_0.

Let $H_0 := \mathcal{H}(\mathbb{Z})$. Then H_0 is isomorphic to the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$, where g_0 represents the generator of $\mathbb{Z}/2\mathbb{Z}$ and where n_0 represents the generator of $\mathbb{Z}$.

Let $V := \mathbb{C}$ and let H_0 act on V by the rule: g_0 acts by multiplication by -1 and n_0 acts trivially.

Let $G_1 \subseteq \mathcal{G}(\mathbb{Q})$ be any subgroup which is commensurable with $\mathcal{G}(\mathbb{Z})$ and which satisfies $G_1 \subseteq H_0$. We will show that the restriction map $H^1(H_0, V) \to H^1(G_1, V)$ is not injective. Note: We may choose G_1 either so that G_1 has connected Zariski closure or so that $H_0 = G_1 N_0$, but not both.

In the exact sequence

$$0 \to H^1(H_0/N_0, V^{N_0}) \to H^1(H_0, V) \to H^1(N_0, V)^{H_0/N_0} \to H^2(H_0/N_0, V^{N_0})$$

the second and fifth terms vanish because H_0/N_0 is finite. Moreover, the map $N_0 \to V$ defined by $n_0 \to 1$ is G_0-equivariant, so

$$H^1(H_0, V) \simeq H^1(N_0, V)^{G_0} \simeq \mathrm{Hom}_{G_0}(N_0, V) \neq 0.$$

On the other hand, G_1 is finite, so $H^1(G_1, V) = 0$. ∎

3. Invariant theory

Let H be a finitely generated group and let G be a finitely generated subgroup of H.

Let V be a finite dimensional complex vector space and let $\eta\colon G \to \mathrm{GL}(V)$ be a (possibly reducible) representation of G on V. Let G act on V by $g.v = \eta(g)v$. Let G act on $\mathrm{gl}\, V$ by $(g.\phi)(v) = g.(\phi(g^{-1}.v))$.

Fix a finite generating set of H. This determines a (possibly reducible) $\mathbb{Q}$-variety of representations of H on V, whose $\mathbb{C}$-points we denote by $\bar{R}$. Let $\mathrm{GL}(V)$ act on $\bar{R}$ by conjugation: if $\tau\colon H \to \mathrm{GL}(V)$ and $a \in \mathrm{GL}(V)$, then $(\tau.a)(h) = a^{-1} \cdot \tau(h) \cdot a$. Invariant theory allows us to form a $\mathbb{Q}$-variety with $\mathbb{C}$-points $\bar{S}$ and $\mathbb{Q}$-morphism $\bar{R} \to \bar{S}$ which makes $\bar{S}$ a universal categorical quotient for the action of $\mathrm{GL}(V)$ on R. (See [LM1, Proposition 1.21, p. 20].)

Let $\bar{R}^s$ denote the principal $\mathbb{Q}$-open subset corresponding to irreducible representations of H on V. Then $\bar{R}^s$ is $\mathrm{GL}(V)$-invariant. Invariant theory shows that the fibers of $\bar{R}^s \to \bar{S}$ are exactly the $\mathrm{GL}(V)$-orbits in $\bar{R}^s$. (Again, see [LM1, Proposition 1.21, p. 20].)

There are analogous $\mathbb{Q}$-varieties $\bar{R}_0$, $\bar{R}_0^s$ and $\bar{S}_0$ for G. There are $\mathbb{Q}$-morphisms $\bar{R} \to \bar{R}_0$ and $\bar{S} \to \bar{S}_0$ defined by restriction of representations from H to G. By abuse of notation, we use η to denote the element of $\bar{R}_0$ corresponding to η. We use $[\eta]$ to denote the image of η under $\bar{R}_0 \to \bar{S}_0$.

The preimage of η under $\bar{R} \to \bar{R}_0$ will be denoted R, the intersection of R and $\bar{R}^s$ by R^s. The preimage of $[\eta]$ under $\bar{S} \to \bar{S}_0$ will be denoted S. Then the $\mathbb{Q}$-morphism $\bar{R} \to \bar{S}$ restricts to a $\mathbb{C}$-morphism $R \to S$. (Since η and $[\eta]$ may not be $\mathbb{Q}$-points, the varieties R and S and the morphism $R \to S$ may not be defined over $\mathbb{Q}$.)

The elements of R correspond to the extensions of η to representations of H. The elements of R^s correspond extensions of η to *irreducible* representations of H. The variety R may be empty and R^s may be empty even if R is not.

Let X denote the kernel of the restriction map $\mathrm{Hom}(H, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$. Then X is the $\mathbb{C}$-points of a linear algebraic $\mathbb{Q}$-group and the identity component X^0 of X is the $\mathbb{C}$-points of a $\mathbb{Q}$-split torus. Then X acts $\mathbb{C}$-algebraically on R by $(\chi.\tau)(h) = \chi(h)\tau(h)$. This action extends to a $\mathbb{Q}$-action of X (or, in fact, of $\mathrm{Hom}(H, \mathbb{C}^*)$) on $\bar{R}$ which commutes with the action of $\mathrm{GL}(V)$. By invariant theory, there is a $\mathbb{Q}$-action of X on $\bar{S}$ so that the map $\bar{R} \to \bar{S}$ is X-equivariant. Since the map $\bar{R} \to \bar{R}_0$ is X-invariant, it follows that the map $\bar{S} \to \bar{S}_0$ is also X-invariant. In particular, X preserves S. As $\bar{R} \to \bar{S}$ is X-equivariant, we conclude that $R \to S$ is X-equivariant as well.

We have a description of the fibers of $R^s \to S$:

LEMMA 3.1: *Let C denote the centralizer in* $\mathrm{GL}(V)$ *of $\eta(G)$. Then R and R^s are C-invariant and the fibers of $R^s \to S$ are exactly the C-orbits in R^s.*

Proof: The C-invariance of R and R^s follows from the definitions.

The group C is the stabilizer in $\mathrm{GL}(V)$ of $\eta \in \bar{R}_0$. Thus, for all $c \in C$, we have $R.c \subseteq R$, whereas, for all $a \in \mathrm{GL}(V)\backslash C$, we have $R.a \cap R = \emptyset$. Since the fibers of $\bar{R}^s \to \bar{S}$ are exactly the $\mathrm{GL}(V)$-orbits in $\bar{R}^s$, we are done. ∎

Notice that, if $\eta(G)$ has reductive Zariski closure, then the C of Lemma 3.1 is reductive and, by invariant theory, we may form the universal categorical quotient of R by C; this would also give S. However, we will not need this fact.

It will be important to have control over the size of the Zariski tangent spaces of points in S. This is provided by:

LEMMA 3.2: *Let $\tau \in R$ have image $[\tau]$ in S. Let H act on V by $h.v = \tau(h)v$ and on $\mathrm{gl}\, V$ by $(h.\phi)(v) = h.(\phi(h^{-1}.v))$. Let K denote the kernel of the restriction map $H^1(H, \mathrm{gl}\, V) \to H^1(G, \mathrm{gl}\, V)$. Then $\dim T_{[\tau]}S \leq \dim K$.*

Proof: By [LM1, Proposition 2.2, p. 33], [LM1, Corollary 2.4, pp. 34-35] and the Luna Slice Theorem, there are injections $T_{[\tau]}\bar{S} \hookrightarrow H^1(H, \mathrm{gl}\, V)$ and $T_{[\eta]}\bar{S}_0 \hookrightarrow H^1(G, \mathrm{gl}\, V)$. The map $T_{[\tau]}\bar{S} \to T_{[\eta]}\bar{S}_0$ extends to the map $H^1(H, \mathrm{gl}\, V) \to H^1(G, \mathrm{gl}\, V)$. Since $T_{[\tau]}S$ is contained in the kernel of $T_{[\tau]}\bar{S} \to T_{[\eta]}\bar{S}_0$, we are done. ∎

LEMMA 3.3: *Let B be a topological group and let A be an open subgroup of finite index in B. Let $S\colon B \to \mathrm{GL}(V)$ be a finite dimensional complex representation of B. Then S is a subrepresentation of $\mathrm{Ind}_A^B(\mathrm{Res}_A^B(S))$.*

Proof: Let T be a set of right coset representatives for A in B. Then T is finite and $B = \coprod_{t \in T} At$.

If T' is any other set of right coset representatives for A in B, then, for all $v \in V$, we have the following equality in $\mathbb{C}[B] \otimes_{\mathbb{C}[A]} V$:

$$\sum_{t \in T} t \otimes t^{-1}v = \sum_{t' \in T'} t' \otimes (t')^{-1}v.$$

It follows that

$$v \mapsto \sum_{t \in T} t \otimes t^{-1}v\colon V \to \mathbb{C}[B] \otimes_{\mathbb{C}[A]} V$$

is a B-equivariant injective linear map. Since $\mathbb{C}[B] \otimes_{\mathbb{C}[A]} V$ is the space of $\mathrm{Ind}_A^B(\mathrm{Res}_A^B(S))$, we are done. ∎

T. Steger pointed out to me the following useful fact:

LEMMA 3.4: *Let B be a topological group and let A be an open subgroup of finite index in B. Then Ind_A^B and Res_A^B both carry completely reducible finite dimensional representations to completely reducible representations.*

Proof: Let $A_0 := \bigcap_{b\in B} bAb^{-1}$ be the normal interior of A in B. Then A_0 is a normal subgroup of finite index in B. Consequently, A_0 is also a normal subgroup of finite index in A.

STEP 1: If A is a *normal* subgroup of B, then Ind_A^B carries completely reducible representations to completely reducible representations.

Proof: Let $R\colon A \to \mathrm{GL}(V)$ be a completely reducible representation of A. Let T be a set of coset representatives for A in B. For all $t \in T$, let $R_t\colon A \to \mathrm{GL}(V)$ be defined by $R_t(a) = R(tat^{-1})$; then R_t is completely reducible and $R' := \mathrm{Res}_A^B(\mathrm{Ind}_A^B(R))$ is isomorphic to $\bigoplus_{t\in T} R_t$, so R' is completely reducible.

Now let W be a B-invariant subspace of $\mathrm{Ind}_A^B(V) := \mathbb{C}[B] \otimes_{\mathbb{C}[A]} V$; we wish to show that W admits a B-invariant complement in $\mathrm{Ind}_A^B(V)$.

Because R' is completely reducible, it follows that there is an A-invariant complement W' to W. Let P denote the projection map $\mathrm{Ind}_A^B(V) = W \oplus W' \to W$. Then P is an A-equivariant idempotent linear map such that $P(V) = W$. Let $P' := \sum_{t\in T} R(t) \circ P \circ R(t^{-1})$. Then P' is a B-equivariant linear map such that $P'(V) \subseteq W$ and such that, for all $w \in W$, we have $P'(w) = w$. Then P' is a B-equivariant idempotent linear map such that $P'(V) = W$. It follows that $\ker(P')$ is a B-invariant complement to W in $\mathrm{Ind}_A^B(V)$.

STEP 2: If A is a *normal* subgroup of B, then Res_A^B carries *irreducible* representations to completely reducible representations.

Proof: Let $S\colon B \to \mathrm{GL}(V)$ be an irreducible representation of B. Let $\mathcal{W}$ be the collection of all A-invariant subspaces of V and let $W \in \mathcal{W}$ have minimal dimension among all elements of $\mathcal{W}$. Then W is irreducible under A. Since A is normal in B, for all $b \in B$, we have: bW is A-invariant and, in fact, irreducible under A. Since V is irreducible under B, it follows that $V = \sum_{b\in B} bW$, so [Jac1, (1)$\Longrightarrow$(2) of Theorem 3.10, p. 121] completes the proof of Step 2.

STEP 3: If A is a *normal* subgroup of B, then Res_A^B carries completely reducible representations to completely reducible representations.

Proof: Since Res_A^B distributes over direct sum, Step 3 follows from Step 2.

STEP 4: In generality, Ind_A^B carries completely reducible representations to completely reducible representations.

Proof: Let $R\colon A \to \mathrm{GL}(V)$ be a completely reducible representation of A. Let $R' := \mathrm{Ind}_{A_0}^A(\mathrm{Res}_{A_0}^A(R))$. By Lemma 3.3, R is a subrepresentation of R', so $\mathrm{Ind}_A^B(R)$ is then a subrepresentation of $\mathrm{Ind}_A^B(R')$. Since a subrepresentation of a completely reducible representation is again completely reducible, it suffices to show that $\mathrm{Ind}_A^B(R')$ is completely reducible. However, $\mathrm{Ind}_A^B(R') = \mathrm{Ind}_{A_0}^B(\mathrm{Res}_{A_0}^A(R))$, so we are done, by Step 1 and Step 3.

STEP 5: In generality, Res_A^B carries completely reducible representations to completely reducible representations.

Proof: Let $S\colon B \to \mathrm{GL}(V)$ be a completely reducible representation of B. Let $R := \mathrm{Res}_A^B(S)$ and let $R_0 := \mathrm{Res}_{A_0}^A(R)$. By Step 1 and Step 3, $R' := \mathrm{Ind}_{A_0}^A(R_0) = \mathrm{Ind}_{A_0}^A(\mathrm{Res}_{A_0}^B(S))$ is completely reducible. By Lemma 3.3, R is a subrepresentation of $\mathrm{Ind}_{A_0}^A(\mathrm{Res}_{A_0}^A(R)) = R'$. Since a subrepresentation of a completely reducible representation is again completely reducible, we are done. ∎

We can now prove that any finite index extension of a completely reducible representation is again completely reducible:

COROLLARY 3.5: *Let B be a topological group and let A be an open subgroup of finite index in B. If $S\colon B \to \mathrm{GL}(V)$ is a finite dimensional complex representation of B and if $\mathrm{Res}_A^B(S)$ is completely reducible, then S is completely reducible as well.*

Proof: By Lemma 3.3, S is a subrepresentation of $S' := \mathrm{Ind}_A^B(\mathrm{Res}_A^B(S))$ and, by Lemma 3.4, S' is completely reducible. Since a subrepresentation of a completely reducible representation is again completely reducible, we are done. ∎

LEMMA 3.6: *Let A and B be locally compact, second countable topological groups. Let $\phi\colon A \to B$ be a continuous homomorphism and assume that $\phi(A)$ is an open subgroup of finite index in B. Let $R\colon A \to \mathrm{GL}(V)$ be a completely reducible finite dimensional complex representation of A. Then there exist (up to isomorphism) only finitely many complex representations $S\colon B \to \mathrm{GL}(V)$ such that $S \circ \phi$ is equivalent to R.*

Proof: By the Baire category theorem, ϕ carries open sets in A to open sets in B.

If $\ker\phi \not\subseteq \ker R$, then no representation $S\colon B \to \mathrm{GL}(V)$ will satisfy: $S \circ \phi$ is equivalent to R. We may therefore assume that $\ker\phi \subseteq \ker R$, i.e. that R factors to a representation $R'\colon \phi(A) \to \mathrm{GL}(V)$. Replacing A by $\phi(A)$ and R by R', we may assume that ϕ is the inclusion of a subgroup A into the group B.

By Lemma 3.4, $S' := \mathrm{Ind}_A^B(R)$ is completely reducible. By Lemma 3.3, any extension of R to a representation on V of B is a subrepresentation of S'. However, a completely reducible represenation has only finitely many isomorphy classes of subrepresentations. ∎

I would be interested to know if Lemma 3.6 remains valid without the assumption that R be completely reducible. The proof breaks down because of the following fact:

Example 3.7: There is a finitely generated group which has a finite dimensional complex representation with uncountably many pairwise non-isomorphic subrepresentations.

Proof: Let

$$\Gamma := \left\{ \begin{bmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; m, n \in \mathbb{Z} \right\}.$$

Let Γ act on $V := \mathbb{C}^{3\times 1}$ by matrix multiplication. For each $x, y, z \in \mathbb{C}$, let $(x,y,z)^t$ denote the 3×1 column matrix whose entries from top to bottom are x, y and z. For each $z \in \mathbb{C}$, define $V_z := \{(x, y, zy)^t \mid x, y \in \mathbb{C}\}$; V_z is Γ-invariant.

Fix $z_1, z_2 \in \mathbb{C}$ and assume that V_{z_1} and V_{z_2} are isomorphic as $\mathbb{C}[\Gamma]$-modules. We wish to show that $z_1 = z_2$.

Let $f\colon V_{z_1} \to V_{z_2}$ be a Γ-equivariant isomorphism. Define $r, s \in \mathbb{C}$ by

$$f((0,1,z_1)^t) = (r, s, z_2 s)^t.$$

Applying

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to $f((0,1,z_1)^t) = (r,s,z_2 s)^t$, we find that $f((1,1,z_1)^t) = (r+s, s, z_2 s)$. Subtracting these last two equations, we have $f((1,0,0)^t) = (s,0,0)^t$. This implies that $s \neq 0$ and that $f((z_1,0,0)^t) = (z_1 s, 0, 0)^t$.

Applying

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to $f((0,1,z_1)^t) = (r,s,z_2s)^t$, we find that $f((z_1,1,z_1)^t) = (r+z_2s,s,z_2s)^t$. Subtracting these last two equations, we have $f((z_1,0,0)^t) = (z_2s,0,0)^t$.

Since we already know that $s \neq 0$ and that $f((z_1,0,0)^t) = (z_1s,0,0)^t$, we find that $z_1 = z_2$, as desired. ∎

I wish to thank S. G. Dani for helping me to find Example 3.7 above.

LEMMA 3.8: *Let B be a countable, discrete group and let A be a subgroup of B. Assume that both A and B are finitely generated. Let $\mathbb{C}$ denote the additive group of complex numbers and let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers. Let K denote the kernel of the restriction map $\mathrm{Hom}(B,\mathbb{C}) \to \mathrm{Hom}(A,\mathbb{C})$ and let K^* denote the kernel of the restriction map $\mathrm{Hom}(B,\mathbb{C}^*) \to \mathrm{Hom}(A,\mathbb{C}^*)$. Then K and K^* are complex Lie groups with the same dimension.*

Proof: The exactness of the exponential sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ implies exactness of

$$\mathrm{Hom}(B,\mathbb{Z}) \to \mathrm{Hom}(B,\mathbb{C}) \to \mathrm{Hom}(B,\mathbb{C}^*).$$

There is a restriction map from this exact sequence to

$$\mathrm{Hom}(A,\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{C}) \to \mathrm{Hom}(A,\mathbb{C}^*).$$

Let K' and Q' denote the kernel and cokernel of the restriction $\mathrm{Hom}(B,\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{Z})$. Applying the Snake Lemma, there is an exact sequence $K' \to K \to K^* \to Q'$ of complex Lie groups. Since K' and Q' are discrete, the result follows. ∎

We can now state the main result of this section:

THEOREM 3.9: *Let $\tau \in R$ have image $[\tau]$ in S. Let H act on V by $h.v = \tau(h)v$ and on $\mathrm{sl}\,V$ by $(h.\phi)(v) = h.(\phi(h^{-1}.v))$. Assume that the restriction map $H^1(H,\mathrm{sl}\,V) \to H^1(G,\mathrm{sl}\,V)$ is injective. Then $X.[\tau]$ is a $\mathbb{C}$-open subset of S consisting entirely of nonsingular points.*

Proof: Let H act on $\mathrm{gl}\,V$ by $(h.\phi)(v) = h.(\phi(h^{-1}.v))$, extending the action on $\mathrm{sl}\,V$. Let I denote the identity element of $\mathrm{gl}\,V$, so that $\mathrm{gl}\,V = \mathrm{sl}\,V \oplus \mathbb{C}I$. By the injectivity assumption, the kernel K of $H^1(H, \mathrm{gl}\, V) \to H^1(G, \mathrm{gl}\, V)$ is isomorphic to the kernel of $H^1(H, \mathbb{C}I) \to H^1(G, \mathbb{C}I)$. The actions of G and H on $\mathbb{C}I$ are trivial, so K is isomorphic to the kernel of $\mathrm{Hom}(H, \mathbb{C}I) \to \mathrm{Hom}(G, \mathbb{C}I)$. By Lemma 3.8, and the definition of X, we have $\dim K = \dim X$. By Lemma 3.2, we can bound the size of the tangent space to S at $[\tau]$: $\dim T_{[\tau]}S \leq \dim X$. Then Theorem 3.9 follows from [Rud1, Lemma 2.1, p. 266] applied to the orbit map $\chi \mapsto \chi.[\tau]\colon X \to S$, which has finite fibers, by [LM1, Lemma 5.8, p. 85]. ∎

We now apply Theorem 3.9 in conjunction with Theorem 2.8 to study representation varieties of arithmetic groups.

COROLLARY 3.10: *Let $\mathcal{H}$ be a linear algebraic $\mathbb{Q}$-group with unipotent radical $\mathcal{N}$. Let $\mathcal{G}$ be a reductive $\mathbb{Q}$-Levi factor of $\mathcal{H}$. Let $G \subseteq \mathcal{G}(\mathbb{Q})$ be commensurable with $\mathcal{G}(\mathbb{Z})$ and let $N \subseteq \mathcal{N}(\mathbb{Q})$ be commensurable with $\mathcal{N}(\mathbb{Z})$. Assume that the Zariski closure of G is connected. Assume that G normalizes N and let $H := GN$. Let η, R^s and X be as defined at the start of §3. Assume that $\eta(G)$ is finite. Let C denote the centralizer in $\mathrm{GL}(V)$ of $\eta(G)$. Let C act on R^s by $(\tau.c)(h) = c^{-1} \cdot \tau(h) \cdot c$. Let X act on R^s by $(\chi.\tau)(h) = \chi(h)\tau(h)$. Then there is a finite subset $F \subseteq R^s$ such that:*

(1) *every C-orbit in R^s intersects $X.F$;*
(2) *if $\tau, \tau' \in F$ and if $\tau \neq \tau'$, then no C-orbit in $X.\tau$ intersects any C-orbit in $X.\tau'$; and*
(3) *for all $\tau \in F$, there is a finite subgroup $X_\tau \subseteq X$ such that, for all $\chi, \chi' \in X$, we have: $\chi.\tau$ is in the C-orbit of $\chi'.\tau$ iff $\chi^{-1}\chi' \in X_\tau$.*

Proof: Let $[R^s]$ denote the image of R^s under the map $R \to S$. Since the map $R \to S$ is X-equivariant, it follows that $[R^s]$ is X-invariant. By [LM1, Lemma 5.8, p. 85], the stabilizer in X of any element of $[R^s]$ is finite. By Lemma 3.1, if we can show that there are only finitely many X-orbits in $[R^s]$, we will be done.

Since the Zariski topology on S is Noetherian, it suffices to show that every X-orbit on S is Zariski open. So fix $\tau \in R^s$ with image $[\tau]$ in S. We wish to show that $X.[\tau]$ is Zariski open.

Let H act on V by $h.v = \tau(h)v$ and on $\mathrm{gl}\, V$ by $(h.\phi)(v) = h.(\phi(h^{-1}.v))$. The H-action on V is irreducible, so $(\mathrm{gl}\, V)^H = \mathbb{C}I$, so $(\mathrm{sl}\, V)^H = 0$. Then by Theorem 2.8, the restriction map $H^1(H, \mathrm{sl}\, V) \to H^1(G, \mathrm{sl}\, V)$ is injective. By Theorem 3.9, we are done. ∎

COROLLARY 3.11: *Let $\mathcal{H}$ be a linear algebraic $\mathbb{Q}$-group with unipotent radical $\mathcal{N}$. Let $G \subseteq \mathcal{G}(\mathbb{Q})$ be commensurable with $\mathcal{G}(\mathbb{Z})$ and let $N \subseteq \mathcal{N}(\mathbb{Q})$ be commensurable with $\mathcal{N}(\mathbb{Z})$. Assume that the Zariski closure of G is connected. Assume that G normalizes N. Let $H := GN$. Let $\tilde{A} := \mathrm{Hom}(N, \mathbb{C}^*)$ be the (noncompact) dual group to the Abelianization of N. Assume that $\tilde{A}^G$ is finite. Then any finite dimensional complex representation of G with finite image has only finitely many irreducible extensions to H, up to isomorphism.*

Proof: The restriction map $X \to \tilde{A}$ takes values in $\tilde{A}^G$, by definition of X. Since $H = GN$, it follows that the map $X \to \tilde{A}^G$ is an isomorphism. Since $\tilde{A}^G$ is finite, X must be finite as well. The result now follows from (1) of Corollary 3.10, since: if two elements of R^s are in the same C-orbit, then they are equivalent representations. ∎

We can improve Corollary 3.11 in that we can avoid the assumption that $H = GN$ and prove that completely reducible extensions are finite in number:

THEOREM 3.12: *Let $\mathcal{H}$ be a linear algebraic $\mathbb{Q}$-group with unipotent radical $\mathcal{N}$. Let $G \subseteq \mathcal{G}(\mathbb{Q})$ be commensurable with $\mathcal{G}(\mathbb{Z})$ and let $N \subseteq \mathcal{N}(\mathbb{Q})$ be commensurable with $\mathcal{N}(\mathbb{Z})$ and let $H \subseteq \mathcal{H}(\mathbb{Q})$ be commensurable with $\mathcal{H}(\mathbb{Z})$. Assume that the Zariski closure of G is connected, that G normalizes N and that $GN \subseteq H$. Let $\tilde{A} := \mathrm{Hom}(N, \mathbb{C}^*)$ be the (noncompact) dual group to the Abelianization of N. Assume that $\tilde{A}^G$ is finite. Then any finite dimensional complex representation of G with finite image has only finitely many completely reducible extensions to H, up to isomorphism.*

Proof: Let $\eta\colon G \to \mathrm{GL}(V)$ be a finite dimensional complex representation of G and assume that $\eta(G)$ is finite. Since η is completely reducible, there is a finite set F of subrepresentations of η such that: every subrepresentation of η is isomorphic to some element of F.

By Corollary 3.11, for each $\rho \in F$, there is a finite set E_ρ of irreducible representations of GN such that: if τ is any irreducible representation of GN which extends ρ, then τ is isomorphic to some element of E_ρ.

Let D be the set of all representations of GN which are obtained by taking a direct sum of at most $\dim V$ elements from $\bigcup_{\rho \in F} E_\rho$. Then D is a finite set of completely reducible representations of GN such that: if $\tau\colon GN \to \mathrm{GL}(V)$ is any completely reducible representation of GN extending η, then τ is isomorphic to some element of D.

As GN has finite index in H, by applying Lemma 3.4, we see that every completely reducible extension of η to H is isomorphic to an extension of some element of D to H. Finally, by Lemma 3.6, each element of D has only finitely many extensions to H, up to isomorphism. ∎

Example 3.13: It is necessary to assume that the Zariski closure of G is connected in Corollary 3.10, in Corollary 3.11 and in Theorem 3.12.

Proof: Let $\mathcal{H}, \mathcal{G}, H_0, G_0, N_0, g_0, n_0, V$ be as in Example 2.10. The representation η of $G_0 = \mathcal{G}(\mathbb{Z})$ on $\mathbb{C}^{2\times 1}$ defined by

$$\eta(g_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has uncountably many pairwise non-isomorphic extensions to irreducible representations of H; indeed, if $\zeta \in \mathbb{C}$, then

$$n_0 \mapsto \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$$

defines an extension of η and, if $\zeta \notin \{\pm 1\}$, then this extension is irreducible. ∎

4. Finite Abelianization

Let $\mathcal{G}$ be a semisimple linear algebraic $\mathbb{Q}$-group. Let $\mathcal{G}^0$ denote the connected component of the identity in $\mathcal{G}$.

Remark 4.1: Any extension of two topological groups with finite Abelianization again has finite Abelianization.

Proof: Assume that A is a normal closed subgroup of B and that both A and B/A have finite Abelianization. Let C be an Abelian topological group and let $f\colon B \to C$ be a surjective continuous homomorphism. We wish to show that C is finite.

As A has finite Abelianization, we know that $f(A)$ is finite. So, by composing $f\colon B \to C$ with the natural map $C \to C/f(A)$, we may assume that $f(A)$ is trivial. Then f factors to B/A which has finite Abelianization. Therefore, the image of f is finite, as desired. ∎

We record two more elementary properties of Abelianization:

Remark 4.2:

(1) A homomorphic image of a group with finite Abelianization again has finite Abelianization.
(2) If a topological group A has a closed subgroup of finite index with finite Abelianization, then A has finite Abelianization. ∎

The converse of item (2) of Remark 4.2 is false: The group $\mathrm{PSL}_2(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ is generated by two elements of finite order; it follows that $\mathrm{PSL}_2(\mathbb{Z})$ has no surjective homomorphisms to $\mathbb{Z}$ and therefore has finite Abelianization. However, $\mathrm{PSL}_2(\mathbb{Z})$ has a free subgroup of finite index. This subgroup then has infinite Abelianization. Another example of this phenomenon comes from the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ which has Abelianization $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but contains an infinite cyclic subgroup of index two.

We can generalize (1) of Remark 4.2:

Remark 4.3: Let $f\colon A \to B$ be a continuous homomorphism of topological groups. Assume that $f(A)$ is dense in B and that A has finite Abelianization. Then B has finite Abelianization. ∎

Remark 4.4: Let A and B be topological groups. Let C be a closed subgroup of finite index in $A \times B$. Then there exist finite index closed subgroups A_0 of A and B_0 of B, such that $A_0 \times B_0 \subseteq C$.

Proof: Let A_0 be the preimage of C under the map $a \mapsto (a,e)\colon A \to A \times B$. Let B_0 be the preimage of C under the map $b \mapsto (e,b)\colon B \to A \times B$. ∎

LEMMA 4.5: *Let k be a local field (i.e., a locally compact, nondiscrete topological field) of characteristic zero. Let $\mathcal{A}$ and $\mathcal{B}$ be k-groups and let $f\colon \mathcal{A} \to \mathcal{B}$ be a surjective k-morphism. Then $f(\mathcal{A}(k))$ is a subgroup of finite index in $\mathcal{B}(k)$.*

Proof: Let $\mathcal{A}$ act on $\mathcal{B}$ by $a.b = f(a)b$. By [BoSe1, Corollaire 6.4, p. 155], the action of $\mathcal{A}(k)$ on $\mathcal{B}(k)$ has finitely many orbits. These orbits are exactly the cosets of $f(\mathcal{A}(k))$ in $\mathcal{B}(k)$. ∎

Lemma 4.5 gives a method to see that a map from one locally compact group to another has finite index image. We will develop a criterion (Lemma 4.7) which allows us to conclude from this that a map from one *discrete* group to another has finite index image.

LEMMA 4.6: *Let A and B be locally compact, second countable topological groups. Let $f\colon A \to B$ be a continuous homomorphism. Let Γ be a lattice in A. Assume that $f(\Gamma)$ is discrete in B. Then $f(A)$ is closed in B and $f(\Gamma)$ is a lattice in $f(A)$.*

Proof: Replacing B by $\overline{f(A)}$, we may assume that $f(A)$ is dense in B. We now wish to show that $f(A) = B$ and that $f(\Gamma)$ is a lattice in B.

Let $g\colon B \to B/f(\Gamma)$ denote the natural map. Let A act on $B/f(\Gamma)$ by $a.x = f(a)x$, for all $a \in A$ and all $x \in B/f(\Gamma)$. There is an A-equivariant map $A/\Gamma \to B/f(\Gamma)$ whose image is $g(f(A))$. Thus $B/f(\Gamma)$ admits an A-invariant probability measure μ concentrated on $g(f(A))$. By the Dominated Convergence Theorem, the stabilizer in B of any probability measure on $B/f(\Gamma)$ is a closed subgroup of B. So, since μ is $f(A)$-invariant, we find that μ is B-invariant. In particular, $f(\Gamma)$ is a lattice in B.

Let ν denote a finite measure in the class of Haar measure on B. Then $g_*(\nu)$ and μ are both in the unique B-quasi-invariant measure class on $B/f(\Gamma)$. Since $g(f(A))$ is μ-conull, it must also be $g_*(\nu)$-conull, which, by definition, means that $g^{-1}(g(f(A)))$ is conull with respect to the Haar measure on B. But $g^{-1}(g(f(A))) = f(A) \cdot f(\Gamma)$ and $f(\Gamma)$ is countable, so $f(A)$ is a subset of B of positive measure. By regularity of Haar measure, $f(A)$ contains a compact subset K such that $\nu(K) > 0$. By [Zim1, Lemma B.4, p. 198], $K^{-1}K$ contains an open neighborhood U of the identity in B. Since $f(A)$ is a subgroup of B, we conclude that $U \subseteq K^{-1}K \subseteq f(A)$. So $f(A)$, being a union of translates of U, is an open subgroup of B. Any open subgroup is closed, since its complement is a union of cosets, all of which are open. So $f(A)$ is closed in B. As $f(A)$ is also dense in B, it follows that $f(A) = B$. ∎

LEMMA 4.7: *Let A and B be locally compact, second countable topological groups. Let Γ be a lattice in A and let Λ be a lattice in B. Let $f\colon A \to B$ be a continouous homomorphism. Assume that $f(\Gamma) \subseteq \Lambda$. Assume that $f(A)$ has finite index in B. Then $f(\Gamma)$ has finite index in Λ.*

Proof: By Lemma 4.6, we have: $f(\Gamma)$ is a lattice in $f(A)$ and $f(A)$ is closed in B. Since $f(A)$ has finite index in B, we conclude that $f(\Gamma)$ is a lattice in B. Since $f(\Gamma) \subseteq \Lambda$ and since Λ is also a lattice in B, it now follows that $f(\Gamma)$ has finite index in Λ. ∎

Definition 4.8: Let A and B be topological groups. We will say that A and B are **abstractly commensurable** if there exist closed finite index subgroups A_0 of A and B_0 of B such that A_0 is isomorphic to B_0.

Recall that $\mathcal{P}$ denotes the set of prime numbers.

LEMMA 4.9: *Assume that $\mathcal{G}$ is almost $\mathbb{Q}$-simple. Let $S_0 \subseteq \mathcal{P}$ be finite. Then there exists a finite subset $S_1 \subseteq \mathcal{P}\backslash S_0$ such that, for all finite $S \subseteq \mathcal{P}$: if $S_1 \subseteq S$, then any group abstractly commensurable with $\mathcal{G}(\mathbb{Z}[S^{-1}])$ has finite Abelianization.*

Proof: By (2) of Remark 4.2, we may replace $\mathcal{G}$ by $\mathcal{G}^0$, and assume that $\mathcal{G}$ is connected.

Choose a finite set $S_1 \subseteq \mathcal{P}\backslash S_0$ such that $\sum_{p\in S_1} \mathrm{rk}_{\mathbb{Q}_p}(\mathcal{G}) \geq 2$. Let S be a finite set satisfying $S_1 \subseteq S \subseteq \mathcal{P}$. Let Γ be abstractly commensurable with $\mathcal{G}(\mathbb{Z}[S^{-1}])$. We wish to show that Γ has finite Abelianization. By (2) of Remark 4.2, we may replace Γ by a subgroup of finite index and assume that Γ is a finite index subgroup of $\mathcal{G}(\mathbb{Z}[S^{-1}])$.

The result now follows from [Mar1, Corollary VIII.2.8(a), p. 266]. ∎

LEMMA 4.10: *Let $S_0 \subseteq \mathcal{P}$ be finite and let $T_0 \subseteq \mathcal{P}\backslash S_0$ be finite. Then there exists a finite $T \subseteq \mathcal{P}\backslash S_0$ such that $T_0 \subseteq T$ and such that: any group abstractly commensurable with $\mathcal{G}(\mathbb{Z}[T^{-1}])$ has finite Abelianization.*

Proof: By (2) of Remark 4.2, we may replace $\mathcal{G}$ by $\mathcal{G}^0$, and assume that $\mathcal{G}$ is connected.

Let $\mathcal{G}_1, \dots, \mathcal{G}_n$ be the almost $\mathbb{Q}$-simple factors of $\mathcal{G}$. Use Lemma 4.9 to choose finite sets $S_1, \dots, S_n \subseteq \mathcal{P}\backslash S_0$ such that, for all $i = 1, \dots, n$, for any finite $S \subseteq \mathcal{P}$, we have:

(∗) if $S_i \subseteq S$, then any group abstractly commensurable with $\mathcal{G}_i(\mathbb{Z}[S^{-1}])$ has finite Abelianization.

Let $T := \left(\bigcup_i S_i\right) \cup T_0$. Let Λ be abstractly commensurable with $\mathcal{G}(\mathbb{Z}[T^{-1}])$. We wish to show that Λ has finite Abelianization.

By (2) of Remark 4.2, we may replace Λ by a subgroup of finite index and assume that Λ is a subgroup of finite index in $\mathcal{G}(\mathbb{Z}[T^{-1}])$.

Let $\mathbb{Q}_\infty := \mathbb{R}$. Let $T^* := T \cup \{\infty\}$. Let ϕ and ψ denote the multiplication maps:

$$\begin{aligned} \phi\colon \quad & \prod_i \mathcal{G}_i(\mathbb{Z}[T^{-1}]) \longrightarrow \mathcal{G}(\mathbb{Z}[T^{-1}]), \\ \psi\colon \quad & \prod_i \prod_{p\in T^*} \mathcal{G}_i(\mathbb{Q}_p) \longrightarrow \prod_{p\in T^*} \mathcal{G}(\mathbb{Q}_p). \end{aligned}$$

By Lemma 4.5, the image of ψ has finite index in $\prod_{p\in T^*} \mathcal{G}(\mathbb{Q}_p)$. We now apply Lemma 4.7 and [Mar1, Theorem I.3.2.5, p. 63] to conclude that the image I of ϕ has finite index in $\mathcal{G}(\mathbb{Z}[T^{-1}])$. By (2) of Remark 4.2, we may replace Λ by $\Lambda \cap I$ and reduce to the case where $\Lambda \subseteq I$. Then $\phi(\phi^{-1}(\Lambda)) = \Lambda$.

Since $\phi^{-1}(\Lambda)$ has finite index in $\prod_i \mathcal{G}_i(\mathbb{Z}[T^{-1}])$, Remark 4.4 repeatedly, then conclude from $(*)$ that $\phi^{-1}(\Lambda)$ has a subgroup of finite index with finite Abelianization. Then, by (2) and (1) of Remark 4.2, we see that Λ has finite Abelianization, as desired. ∎

COROLLARY 4.11: *For all $p \in \mathcal{P}$, any group abstractly commensurable with $\mathcal{G}(\mathbb{Z}_p)$ has finite Abelianization.*

Proof: Assume that A is abstractly commensurable with $\mathcal{G}(\mathbb{Z}_p)$. We wish to show that A has finite Abelianization.

By (2) of Remark 4.2, we may replace A by a closed subgroup of finite index and assume that A is a closed subgroup of finite index in $\mathcal{G}^0(\mathbb{Z}_p)$. In particular, A is a compact open subgroup of $\mathcal{G}^0(\mathbb{Q}_p)$.

Let $\tilde{\mathcal{G}}$ denote the simply connected covering of $\mathcal{G}^0$. Let $\tilde{A}$ denote the preimage in $\tilde{\mathcal{G}}(\mathbb{Q}_p)$ of A. Since the projection map $\pi\colon \tilde{\mathcal{G}}(\mathbb{Q}_p) \to \mathcal{G}^0(\mathbb{Q}_p)$ is continuous and finite-to-one, it follows that $\tilde{A}$ is a compact open subgroup of $\tilde{\mathcal{G}}(\mathbb{Q}_p)$. Any two compact open subgroups are commensurable, so $\tilde{A}$ is commensurable with $\tilde{\mathcal{G}}(\mathbb{Z}_p)$. By Lemma 4.5, $\pi(\tilde{\mathcal{G}}(\mathbb{Q}_p))$ has finite index in $\mathcal{G}^0(\mathbb{Q}_p)$, so $\pi(\tilde{A}) = A \cap \pi(\tilde{\mathcal{G}}(\mathbb{Q}_p))$ has finite index in A. So, by (1) and (2) of Remark 4.2, we may replace A by $\tilde{A}$ and $\mathcal{G}$ by $\tilde{\mathcal{G}}$ to assume that $\mathcal{G}$ is connected and algebraically simply connected.

Now choose $q \in \mathcal{P}\backslash p$ such that $\mathrm{rk}_{\mathbb{Q}_q}(\mathcal{G}) > 0$. Let $S_0 := \{p\}$ and $T_0 := \{q\}$ and choose T as in Lemma 4.10. Then $p \notin T$, $q \in T$ and

$(*)$ any group abstractly commensurable with $\mathcal{G}(\mathbb{Z}[T^{-1}])$ has finite Abelianization.

By strong approximation, the image of $\mathcal{G}(\mathbb{Z}[T^{-1}])$ in $\mathcal{G}(\mathbb{Z}_p)$ is dense.

By (2) of Remark 4.2, we may replace A by a closed subgroup of finite index and assume that A is a closed subgroup of finite index in $\mathcal{G}(\mathbb{Z}_p)$. Then the preimage Γ of A in $\mathcal{G}(\mathbb{Z}[T^{-1}])$ has finite index in $\mathcal{G}(\mathbb{Z}[T^{-1}])$. By $(*)$, Γ has finite Abelianization. But A is open and closed in $\mathcal{G}(\mathbb{Z}_p)$, so A is the closure of the image of Γ in $\mathcal{G}(\mathbb{Z}_p)$. By Remark 4.3, A has finite Abelianization, as desired. ∎

COROLLARY 4.12: *If $\mathcal{G}$ is connected and algebraically simply connected, then any group abstractly commensurable with $\prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$ has finite Abelianization.*

Proof: Let A be abstractly commensurable with $\prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. We wish to show that A has finite Abelianization.

Let $q \in \mathcal{P}$ satisfy: every almost $\mathbb{Q}$-simple factor of $\mathcal{G}$ is $\mathbb{Q}_q$-isotropic. Let $S_0 := \emptyset$ and $T_0 := \{q\}$. Choose a finite set $T \subseteq \mathcal{P}$, as in Lemma 4.10. Then $q \in T$ and

$(*)$ any group abstractly commensurable with $\mathcal{G}(\mathbb{Z}[T^{-1}])$ has finite Abelianization.

By strong approximation, the image of the diagonal map

$$\psi\colon \mathcal{G}(\mathbb{Z}[T^{-1}]) \to \prod_{p\in\mathcal{P}\setminus T} \mathcal{G}(\mathbb{Z}_p)$$

is dense. Let

$$\pi\colon \prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p) \to \prod_{p\in\mathcal{P}\setminus T} \mathcal{G}(\mathbb{Z}_p)$$

be the projection map. By (2) of Remark 4.2, we may pass to a closed subgroup of finite index in A and assume that A is a closed subgroup of finite index in $\prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. Then A is an extension of $L := A \cap (\ker \pi)$ and $A_1 := \pi(A)$. By Remark 4.1, it suffices to show that L and A_1 have finite Abelianization.

Now L is (isomorphic to) a closed subgroup of finite index in $\prod_{p\in T} \mathcal{G}(\mathbb{Z}_p)$. By repeated applications of Remark 4.4, we may, for each $p \in T$, choose a finite index closed subgroup L_p of $\mathcal{G}(\mathbb{Z}_p)$ such that: some finite index closed subgroup of L is isomorphic to $\prod_{p\in T} L_p$. By Corollary 4.11, for all $p \in T$, L_p has finite Abelianization. Then $\prod_{p\in T} L_p$ has finite Abelianization, so, by (2) of Remark 4.2, L has finite Abelianization. It remains to show that A_1 has finite Abelianization.

Since A is a compact open subgroup in $\prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$, it follows that $A_1 = \pi(A)$ is a compact open subgroup in $\prod_{p\in\mathcal{P}\setminus T} \mathcal{G}(\mathbb{Z}_p)$. Since the diagonal embedding ψ has dense image, it follows that A_1 is the closure of $\psi(\psi^{-1}(A_1))$. Now A_1 has finite index in $\prod_{p\in\mathcal{P}\setminus T} \mathcal{G}(\mathbb{Z}_p)$, so $\psi^{-1}(A_1)$ has finite index in $\mathcal{G}(\mathbb{Z}[T^{-1}])$.

By $(*)$, $\psi^{-1}(A_1)$ has finite Abelianization. So, by Remark 4.3, the closure in $\prod_{p\in\mathcal{P}\setminus T}\mathcal{G}(\mathbb{Z}_p)$ of $\psi(\psi^{-1}(A_1))$ has finite Abelianization. That is, A_1 has finite Abelianization, as desired. ∎

5. Congruence representations

Let $m > 0$ be a integer. Let $\mathcal{H}$ be a $\mathbb{Q}$-subgroup of SL_m. Let $\mathcal{N}$ denote the unipotent radical of $\mathcal{H}$. For all integers $K > 0$, let $\mathrm{Rd}_K\colon \mathcal{H}(\mathbb{Z}) \to \mathcal{H}(\mathbb{Z}/K\mathbb{Z})$ denote reduction modulo K. For all integers $K > 0$, define $\mathcal{H}(K) := \ker(\mathrm{Rd}_K)$.

LEMMA 5.1: *Let $\mathcal{G}$ be a $\mathbb{Q}$-Levi subgroup of $\mathcal{H}$. There exists an integer $m' > 0$, a $\mathbb{Q}$-subgroup $\mathcal{H}' \subseteq \mathrm{SL}_{m'}$ and a $\mathbb{Q}$-isomorphism $\pi\colon \mathcal{H}' \to \mathcal{H}$ such that, if we define*

(A) *$\mathcal{G}' := \pi^{-1}(\mathcal{G})$ and $\mathcal{N}' := \pi^{-1}(\mathcal{N})$; and*

(B) *for all integers $K > 0$, $\mathcal{H}'(K)$ is the kernel of the map $\mathcal{H}'(\mathbb{Z}) \to \mathcal{H}'(\mathbb{Z}/K\mathbb{Z})$ given by reduction modulo K,*

then we have

(1) *$\mathcal{H}'(\mathbb{Z}) = \mathcal{G}'(\mathbb{Z})\mathcal{N}'(\mathbb{Z})$;*

(2) *$\pi(\mathcal{G}'(\mathbb{Z})) = \mathcal{G}(\mathbb{Z})$ and $\pi(\mathcal{N}'(\mathbb{Z})) = \mathcal{N}(\mathbb{Z})$; and*

(3) *for all integers $K > 0$, $\pi(\mathcal{H}'(K)) \subseteq \mathcal{H}(K)$.*

Proof: Let $p\colon \mathcal{H} \to \mathcal{G}$ denote the projection modulo the unipotent radical. This defines a map $h \mapsto (h, p(h))\colon \mathcal{H} \to \mathcal{H} \times \mathcal{G} \subseteq \mathrm{SL}_m \times \mathrm{SL}_m$.

Let $m' := 2m$ and embed $\mathrm{SL}_m \times \mathrm{SL}_m \subseteq \mathrm{SL}_{m'}$. Let $\mathcal{H}' \subseteq \mathrm{SL}_{m'}$ denote the image of $\mathcal{H}$. Let $\pi\colon \mathcal{H}' \to \mathcal{H}$ denote the restriction of the projection map $\mathcal{H} \times \mathcal{G} \to \mathcal{H}$. ∎

Definition 5.2: A **congruence representation for $\mathcal{H}$** is a finite dimensional complex representation $\tau\colon \mathcal{H}(\mathbb{Z}) \to \mathrm{GL}(V)$ such that, for some integer $K > 0$, $\mathcal{H}(K) \subseteq \ker\tau$.

LEMMA 5.3: *Assume that $\mathcal{H}$ is connected and algebraically simply connected and that every almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic. Let K and L be positive integers. Then $\mathcal{H}(K)\mathcal{H}(L) = \mathcal{H}(\gcd(K, L))$.*

Proof: We have $\mathcal{H}(K), \mathcal{H}(L) \subseteq \mathcal{H}(\gcd(K, L))$. Fix $z \in \mathcal{H}(\gcd(K, L))$. We wish to find $x \in \mathcal{H}(K)$ and $y \in \mathcal{H}(L)$ such that $z = xy$.

Let S denote the set of primes which divide KL. By strong approximation, the diagonal embedding $\phi\colon \mathcal{H}(\mathbb{Z}) \to \prod_{p\in S}\mathcal{H}(\mathbb{Z}_p)$ has dense image.

For all $p \in S$, let

$$e_p := \max\{e\colon p^e|K\}, \quad f_p := \max\{f\colon p^f|L\}, \quad g_p := \max\{g\colon p^g|\gcd(K,L)\}.$$

Then, for all $p \in S$, we have $g_p = \min\{e_p, f_p\}$. For all $p \in S$, let U_p, V_p, W_p be the kernels of

$$\mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}/p^{e_p}\mathbb{Z}), \quad \mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}/p^{f_p}\mathbb{Z}), \quad \mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}/p^{g_p}\mathbb{Z}).$$

Then, for all $p \in S$, either $U_p \subseteq V_p$ or $V_p \subseteq U_p$, so $W_p = U_p \cup V_p = U_p V_p$.

Fix $p \in S$. Let z_p denote the image of z in $\mathcal{H}(\mathbb{Z}_p)$. Then $z_p \in W_p = U_p V_p$. Choose $\tilde{x}_p \in U_p$ and $\tilde{y}_p \in V_p$ such that $z_p = \tilde{x}_p \tilde{y}_p$. So $\tilde{x}_p^{-1} z_p \in V_p$. Since U_p and V_p are open in $\mathcal{H}(\mathbb{Z}_p)$, it follows that there is a neighborhood $\tilde{U}_p$ of $\tilde{x}_p$ such that: $\tilde{U}_p \subseteq U_p$ and $\tilde{U}_p^{-1} z_p \subseteq V_p$.

Since the image of ϕ is dense, we may choose $x \in \mathcal{H}(\mathbb{Z})$ such that, for all $p \in S$, the image x_p of x in $\mathcal{H}(\mathbb{Z}_p)$ satisfies: $x_p \in \tilde{U}_p$. Let $y := x^{-1}z$. We wish to show that $x \in \mathcal{H}(K)$ and that $y \in \mathcal{H}(L)$.

For all $p \in S$, let y_p denote the image of y in $\mathcal{H}(\mathbb{Z}_p)$. For all $p \in S$, we have

$$x_p \in \tilde{U}_p \subseteq U_p, \qquad y_p = x_p^{-1} z_p \in \tilde{U}_p^{-1} z_p \subseteq V_p.$$

So

$$x \in \phi^{-1}\left(\prod_{p \in S} U_p\right) = \mathcal{H}(K), \qquad y \in \phi^{-1}\left(\prod_{p \in S} V_p\right) = \mathcal{H}(L),$$

as desired. ∎

COROLLARY 5.4: *Assume that $\mathcal{H}$ is connected and algebraically simply connected. Assume moreover that no almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic. Let τ denote a congruence representation for $\mathcal{H}$. Then there exists an integer $K_\tau > 0$ such that, for all integers $K > 0$, we have: $\mathcal{H}(K) \subseteq \ker\tau \iff K_\tau|K$.*

Proof: Let $K_\tau := \min\{K \in \mathbb{N} \,|\, \mathcal{H}(K) \subseteq \ker\tau\}$. Then

$$K_\tau|K \Longrightarrow \mathcal{H}(K) \subseteq \mathcal{H}(K_\tau) \subseteq \ker\tau.$$

Fix an integer $K > 0$ and assume that $\mathcal{H}(K) \subseteq \ker\tau$. We wish to show that $K_\tau|K$.

By Lemma 5.3, $\mathcal{H}(\gcd(K_\tau, K)) = \mathcal{H}(K_\tau)\mathcal{H}(K) \subseteq \ker\tau$. Then,

$$K_\tau \le \gcd(K_\tau, K),$$

by minimality. This implies that $K_\tau|K$. ∎

Remark 5.5: Let $K, K' > 0$ be integers. Then there exist integers $A, B, A', B' > 0$ such that:

(1) $K = AB$, $K' = A'B'$;
(2) $A|A'$, $B'|B$;
(3) $\operatorname{lcm}(K, K') = A'B$, $\gcd(K, K') = AB'$; and
(4) A' and B are relatively prime.

Proof: Let $S \subseteq \mathcal{P}$ denote the set of all prime numbers dividing KK'. For all $p \in S$, define

$$e_p := \max\{e \in \mathbb{N} \colon p^e|K\}, \qquad e'_p := \max\{e' \in \mathbb{N} \colon p^{e'}|K'\}.$$

Let $T := \{p \in S : e'_p \geq e_p\}$ and $U := S \backslash T$. Define

$$A := \prod_{p \in T} p^{e_p}, \qquad B := \prod_{p \in U} p^{e_p}, \qquad A' := \prod_{p \in T} p^{e'_p}, \qquad B' := \prod_{p \in U} p^{e'_p}. \qquad \blacksquare$$

LEMMA 5.6: *Assume that $\mathcal{H}$ is algebraically simply connected and that every almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic. Let $\mathcal{G}$ be a $\mathbb{Q}$-Levi factor of $\mathcal{H}$. Let X denote the kernel of the restriction map $\operatorname{Hom}(\mathcal{H}(\mathbb{Z}), \mathbb{C}^*) \to \operatorname{Hom}(\mathcal{G}(\mathbb{Z}), \mathbb{C}^*)$. Let V be a finite dimensional complex vector space. Let $\tau \colon \mathcal{H}(\mathbb{Z}) \to \operatorname{GL}(V)$ be a representation. Fix $\chi \in X$ and fix two integers $K, K' > 0$. Assume that*

$$\mathcal{H}(K) \subseteq \ker(\tau) \qquad \text{and} \qquad \mathcal{H}(K') \subseteq \ker(\chi.\tau).$$

Then there exists $\psi \in X$ such that $\mathcal{H}(\gcd(K, K')) \subseteq \ker(\psi.\tau)$.

Proof: Let $H := \mathcal{H}(\mathbb{Z})$, $G := \mathcal{G}(\mathbb{Z})$. Choose integers $A, B, A', B' > 0$ as in Remark 5.5.

Fix an integer $L > 0$. Let $\mathcal{H}(L)$ and H_L denote, respectively, the kernel and image of the mod L reduction map $\mathcal{H}(\mathbb{Z}) \to \mathcal{H}(\mathbb{Z}/L\mathbb{Z})$. Similarly, let $\mathcal{G}(L)$ and G_L denote, respectively, the kernel and image of the mod L reduction map $\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/L\mathbb{Z})$. There is a natural embedding of G_L into H_L.

Now $\mathcal{H}(A'B) \subseteq \ker(\tau) \cap \ker(\chi.\tau) \subseteq \ker(\chi)$, so χ factors to a character $\bar{\chi}$ on $H_{A'B}$. Since A' and B are relatively prime, strong approximation implies that the natural maps $H_{A'B} \to H_{A'} \times H_B$ and $G_{A'B} \to G_{A'} \times G_B$ are isomorphisms.

Because of these isomorphisms, the maps $H_{A'B} \to H_{A'}$ and $H_{A'B} \to H_B$ are identified with the projections $H_{A'} \times H_B \to H_{A'}$ and $H_{A'} \times H_B \to H_B$.

The character $\bar{\chi}$ on $H_{A'B}$ becomes identified with a character $\tilde{\chi}$ on $H_{A'} \times H_B$. The embedding of $G_{A'B}$ into $H_{A'B}$ becomes identified with the embedding of $G_{A'} \times G_B$ in $H_{A'} \times H_B$. Thus $\tilde{\chi}|G_{A'} \times G_B$ is trivial.

Any character of $H_{A'} \times H_B$ can be written as a tensor product of a character on $H_{A'}$ by a character on H_B. In particular, there exist characters $\tilde{\phi}$ on $H_{A'}$ an $\tilde{\psi}$ on H_B such that, for all $(h, h') \in H_{A'} \times H_B$, we have $\tilde{\chi}(h,h') = \tilde{\phi}(h)\tilde{\psi}(h')$. Let ϕ be the composition of $\tilde{\phi}$ with $H \to H_{A'}$ and let ψ be the composition of $\tilde{\psi}$ with $H \to H_B$. Then

(i) ϕ factors to a character on $H_{A'}$ whose restriction to $G_{A'}$ is trivial;
(ii) ψ factors to a character on H_B whose restriction to G_B is trivial; and
(iii) $\chi = \phi\psi$, i.e., for all $h \in H$, we have $\chi(h) = \phi(h)\psi(h)$.

By (i) and (ii), we have that $\phi, \psi \in X$, that $\mathcal{H}(A') \subseteq \ker(\phi)$ and that $\mathcal{H}(B) \subseteq \ker(\psi)$.

We know $\mathcal{H}(K) \subseteq \ker(\tau)$ and $\mathcal{H}(K') \subseteq \ker(\chi.\tau)$. We want $\mathcal{H}(\gcd(K, K')) \subseteq \ker(\psi.\tau)$.

Now $\mathcal{H}(K) \subseteq \ker(\tau)$ and $\mathcal{H}(K) \subseteq \mathcal{H}(B) \subseteq \ker(\psi)$, so $\mathcal{H}(K) \subseteq \ker(\psi.\tau)$. Moreover, since $\mathcal{H}(K') \subseteq \ker(\chi.\tau) = \ker((\phi\psi).\tau)$ and $\mathcal{H}(K') \subseteq \mathcal{H}(A') \subseteq \ker(\phi)$, we conclude that $\mathcal{H}(K') \subseteq \ker(\psi.\tau)$.

Thus, by Lemma 5.3, we have $\mathcal{H}(\gcd(K, K')) = \mathcal{H}(K)\mathcal{H}(K') \subseteq \ker(\psi.\tau)$, as desired. ∎

LEMMA 5.7: *Let A be a topological group with a dense finitely generated subgroup. Let $I > 0$ be an integer. Then A has only finitely many open subgroups of index I.*

Proof: Assume that A has a dense subgroup that is generated by k elements. Let $\mathbb{F}_k$ denote the free group on k letters. Then there is a homomorphism $h\colon \mathbb{F}_k \to A$ such that $h(\mathbb{F}_k)$ is dense in A.

If A_0 is an open subgroup of A of index I, then $h^{-1}(A_0)$ is a subgroup of $\mathbb{F}_k$, also of index I. If A_0, A_1 are open subgroups of A, then: $h^{-1}(A_0) = h^{-1}(A_1)$ iff $A_0 = A_1$. We may therefore assume that $A = \mathbb{F}_k$.

Let X denote the graph with one vertex and k edges; then $A = \pi_1(X)$. The subgroups of A of index I are in one-to-one correspondence with I-fold covering graphs of X with distinguished vertex. These are finite in number. ∎

Lemma 5.7 is an observation of J.-P. Serre. The next result is also due to Serre, who recognized that it represented the appropriate level of generality of a similar argument of Z. Rudnick. (See [SarAd1, Rudnick's appendix].) Both of these results appear in an unpublished letter from Serre to P. Sarnak. The following statement and proof are taken almost *verbatim* from this letter. Throughout this proof, if U is any profinite group, then U^{ab} denotes the quotient of U by the closure of its commutator subgroup, i.e., U^{ab} denotes the Abelianization of U. Lemma 3.3 is used implicitly without comment in the proof of the (b)$\Longrightarrow$(a) part of Theorem 5.8. For an explanation of "Jordan bound", see [CR1, Theorem 36.13, p. 258].

THEOREM 5.8: *Let P be a profinite group. The following are equivalent:*

(a) *For every $n \geq 1$, there are only finitely many isomorphy classes of continuous linear representations $P \to \mathrm{GL}_n(\mathbb{C})$.*

(b) *For every open subgroup U of P, the group U^{ab} is finite; moreover P has only finitely many open subgroups of a given (finite) index.*

Proof: (a)$\Longrightarrow$(b). If P had infinitely many open subgroups U of the same index n, this would give (by inducing the 1 representation of U) infinitely many distinct representations of P of degree n. Similarly, if an open subgroup U of index n is such that U^{ab} is infinite, this gives infinitely many characters of U and again by induction infinitely many distinct representations of P of degree n.

(b)$\Longrightarrow$(a). Let $j(n)$ denote the Jordan bound with respect to n. By (b), P has only finitely many open subgroups U of index $\leq j(n)$, and each of them has only finitely many one-dimensional characters. By induction this gives finitely many representations of P, and every irreducible representation of P of degree n is a factor of one of these. ∎

LEMMA 5.9: *Let $\mathcal{G}$ be a connected, algebraically simply connected, semisimple linear algebraic $\mathbb{Q}$-group. Then $\prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$ contains a dense finitely generated subgroup.*

Proof: Choose $q \in \mathcal{P}$ such that $\mathrm{rk}_{\mathbb{Q}_q}(\mathcal{G}) > 0$. Let $\mathcal{A}$ be the product of the $\mathbb{R}$-isotropic almost $\mathbb{R}$-simple factors of $\mathcal{G}$. Let $\mathcal{B}$ be the product of the $\mathbb{Q}_q$-isotropic almost $\mathbb{Q}_q$-simple factors of $\mathcal{G}$.

By [Mar1, Theorem I.3.2.5, p. 63], the diagonal image of $\mathcal{G}(\mathbb{Z}[1/q])$ in $\mathcal{G}(\mathbb{R}) \times \mathcal{G}(\mathbb{Q}_q)$ is a lattice. The projection map $\mathcal{G}(\mathbb{R}) \times \mathcal{G}(\mathbb{Q}_q) \to \mathcal{A}(\mathbb{R}) \times \mathcal{B}(\mathbb{Q}_q)$ has compact kernel, so the image I of $\mathcal{G}(\mathbb{Z}[1/q])$ in $\mathcal{A}(\mathbb{R}) \times \mathcal{B}(\mathbb{Q}_q)$ is a lattice which, by [Mar1, Remark (i), p. 289], has property (QD). Then, by [Mar1, Theorem IX.3.2(i), p. 312], I is finitely generated.

There is a finite normal subgroup K of $\mathcal{G}(\mathbb{Z}[1/q])$ such that $\mathcal{G}(\mathbb{Z}[1/q])/K \cong I$, so it follows that $\mathcal{G}(\mathbb{Z}[1/q])$ is finitely generated. By strong approximation, the diagonal image D_0 of $\mathcal{G}(\mathbb{Z}[1/q])$ in $\prod_{p\in\mathcal{P}\backslash q}\mathcal{G}(\mathbb{Z}_p)$ is dense.

Now choose $r \in \mathcal{P}\backslash q$ such that $\mathrm{rk}_{\mathbb{Q}_r}(\mathcal{G}) > 0$. By the same argument, $\prod_{p\in\mathcal{P}\backslash r}\mathcal{G}(\mathbb{Z}_p)$ contains a dense finitely generated subgroup. The projection $\prod_{p\in\mathcal{P}\backslash r}\mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}_q)$ is surjective, so $\mathcal{G}(\mathbb{Z}_q)$ contains a dense finitely generated subgroup D_1.

Then $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p) \cong \mathcal{G}(\mathbb{Z}_q) \times \prod_{p\in\mathcal{P}\backslash q}\mathcal{G}(\mathbb{Z}_p)$ contains the dense finitely generated subgroup $D_1 \times D_0$. ∎

The next lemma is straightforward.

Lemma 5.10: *If $1 \to A \to B \to C \to 1$ is an exact sequence of topological groups and if A and C contain dense finitely generated subgroups, then B does as well.*

Proposition 5.11: *If $\mathcal{H}$ is connected and algebraically simply connected, then $\prod_{p\in\mathcal{P}}\mathcal{H}(\mathbb{Z}_p)$ contains a dense finitely generated subgroup.*

Proof: Let $\mathcal{G} := \mathcal{H}/\mathcal{N}$. Since $\mathcal{H}$ is connected and algebraically simply connected, it follows that $\mathcal{G}$ is connected, algebraically simply connected and semisimple. The projection map $\pi: \mathcal{H} \to \mathcal{G}$ has a right inverse $\iota: \mathcal{G} \to \mathcal{H}$ defined over $\mathbb{Q}$. The image of $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$ under ι is a compact subgroup of the adelic points $\mathcal{H}(\mathbb{A})$ of $\mathcal{H}$. Since $B := \prod_{p\in\mathcal{P}}\mathcal{H}(\mathbb{Z}_p)$ is open in $\mathcal{H}(\mathbb{A})$, it follows that there exists a finite index open subgroup V of $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$ such that $\iota(V) \subseteq B$. Let $C := \pi(B)$. Then $V \subseteq C$. Thus C is a compact open subgroup of $\mathcal{G}(\mathbb{A})$ and is therefore commensurable with $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$. Let $U := C \cap \prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$.

It follows from Lemma 5.9 that $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$ contains a dense finitely generated subgroup D. Since a finite index subgroup of a finitely generated group is again finitely generated, it follows that $U \cap D$ is finitely generated. On the other hand, U is open in $\prod_{p\in\mathcal{P}}\mathcal{G}(\mathbb{Z}_p)$, so $U \cap D$ is dense in U.

So, since U contains a dense finitely generated subgroup, and since U has finite index in C, we conclude that C contains a dense finitely generated subgroup as well. Then, by Lemma 5.10, it suffices to show that $A := \prod_{p\in\mathcal{P}}\mathcal{N}(\mathbb{Z}_p)$ contains a dense finitely generated subgroup.

By strong approximation, the diagonal image of $\mathcal{N}(\mathbb{Z})$ in A is dense, so it suffices to show that $\mathcal{N}(\mathbb{Z})$ is finitely generated. This follows from [Rag1, Theorem 2.10, p. 32]. ∎

THEOREM 5.12: *Let $m, \nu > 0$ be integers. Let $\mathcal{G}$ be a semisimple $\mathbb{Q}$-subgroup of SL_m. Then $\mathcal{G}$ has only finitely many ν-dimensional congruence representations, up to isomorphism.*

Proof: Let $\pi: \tilde{\mathcal{G}} \to \mathcal{G}^0$ denote the simply connected covering of the connected component $\mathcal{G}^0$ of the identity in $\mathcal{G}$. Choose an integer $m'' > 0$ and an embedding ι of $\tilde{\mathcal{G}}$ into $\mathrm{SL}_{m''}$. Let $m' := m + m''$. Embed $\tilde{\mathcal{G}}$ into $\mathrm{SL}_m \times \mathrm{SL}_{m''} \subseteq \mathrm{SL}_{m'}$ by $\tilde{g} \mapsto (\pi(\tilde{g}), \iota(\tilde{g}))$. Use this embedding to define $\tilde{\mathcal{G}}(\mathbb{Z})$, and, for any integer $K > 0$, to define the Kth congruence subgroup $\tilde{\mathcal{G}}(K)$. Then $\pi(\tilde{\mathcal{G}}(\mathbb{Z})) \subseteq \mathcal{G}(\mathbb{Z})$ and, for any integer $K > 0$, we have: $\pi(\tilde{\mathcal{G}}(K)) \subseteq \mathcal{G}(K)$.

By [Mar1, Theorem I.3.2.8(a), p. 64], $\tilde{\mathcal{G}}(\mathbb{Z})$ is a lattice in $\tilde{\mathcal{G}}(\mathbb{R})$ and $\mathcal{G}(\mathbb{Z})$ is a lattice in $\mathcal{G}(\mathbb{R})$. Further, $\pi(\tilde{\mathcal{G}}(\mathbb{R}))$ has finite index in $\mathcal{G}(\mathbb{R})$, by Lemma 4.5. Thus, by Lemma 4.7, we conclude that $\pi(\tilde{\mathcal{G}}(\mathbb{Z}))$ has finite index in $\mathcal{G}(\mathbb{Z})$. We may therefore apply Lemma 3.6 to conclude that the pullback map from finite image representations of $\mathcal{G}(\mathbb{Z})$ to finite image representations of $\tilde{\mathcal{G}}(\mathbb{Z})$ is finite-to-one. Finally, because of our requirements on the embedding of $\tilde{\mathcal{G}}$ in $\mathrm{SL}_{m'}$, we see that the pullback of a congruence representation for $\mathcal{G}$ is a congruence representation for $\tilde{\mathcal{G}}$. Therefore, it suffices to show that $\tilde{\mathcal{G}}$ has only finitely many ν-dimensional congruence representations, up to isomorphism. We may therefore replace $\mathcal{G}$ by $\tilde{\mathcal{G}}$ and assume that $\mathcal{G}$ is connected and algebraically simply connected.

Let $\mathcal{G}'$ denote the product of the $\mathbb{R}$-isotropic almost $\mathbb{Q}$-simple factors of $\mathcal{G}$. By [Mar1, Theorem I.3.2.8(a), p. 64], $\mathcal{G}'(\mathbb{Z})$ is a lattice in $\mathcal{G}'(\mathbb{R})$ and $\mathcal{G}(\mathbb{Z})$ is a lattice in $\mathcal{G}(\mathbb{R})$. Moreover, $\mathcal{G}(\mathbb{R})/\mathcal{G}'(\mathbb{R})$ is compact. So, by [Rag1, Lemma 1.6, p. 20], $\mathcal{G}'(\mathbb{Z})$ is a lattice in $\mathcal{G}(\mathbb{R})$. Since $\mathcal{G}(\mathbb{Z})$ is also a lattice in $\mathcal{G}(\mathbb{R})$, and since $\mathcal{G}'(\mathbb{Z}) \subseteq \mathcal{G}(\mathbb{Z})$, we conclude that $\mathcal{G}'(\mathbb{Z})$ has finite index in $\mathcal{G}(\mathbb{Z})$. Moreover, the restriction to $\mathcal{G}'(\mathbb{Z})$ of a congruence representation for $\mathcal{G}$ is a congruence representation for $\mathcal{G}'$. We may therefore apply Lemma 3.6 again to see that it suffices to show that $\mathcal{G}'$ has only finitely many ν-dimensional congruence representations. Replacing $\mathcal{G}$ by $\mathcal{G}'$, we now assume that every almost $\mathbb{Q}$-simple factor of $\mathcal{G}$ is $\mathbb{R}$-isotropic.

Let $\rho: \mathcal{G}(\mathbb{Z}) \to \prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$ be the diagonal embedding. By strong approximation, we know that $\rho(\mathcal{G}(\mathbb{Z}))$ is dense in $\prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. Further, every congruence subgroup $\mathcal{G}(K)$ for $\mathcal{G}$ is the ρ-preimage of some compact open subgroup of $\prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. Thus, every congruence representation for $\mathcal{G}$ arises by composing ρ with some continuous representation of $\prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. Moreover, by density, no two continuous representations of $\prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$ become the same on precomposing with ρ. We are therefore reduced to showing that $\prod_{p \in \mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$ has only finitely many continuous ν-dimensional representations.

By Lemma 5.9, Lemma 5.7 and Corollary 4.12, both conditions in (b) of Theorem 5.8 are verified for $P := \prod_{p\in\mathcal{P}} \mathcal{G}(\mathbb{Z}_p)$. By (a) of Theorem 5.8, we are done. ∎

THEOREM 5.13: *Assume that $\mathcal{H}/\mathcal{N}$ is semisimple. Let $\mathcal{G}$ be a $\mathbb{Q}$-Levi factor of $\mathcal{H}$. Suppose that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$ and that the Zariski closure of $\mathcal{G}(\mathbb{Z})$ is connected. Let X_c denote the set of all one-dimensional congruence representations for $\mathcal{H}$ whose restriction to $\mathcal{G}(\mathbb{Z})$ is trivial. Fix an integer $\nu > 0$. Let R_c^s denote the set of isomorphy classes of irreducible ν-dimensional congruence representations for $\mathcal{H}$. Let X_c act on R_c^s by $(\chi.\tau)(h) = \chi(h)\tau(h)$. Then there exists a finite subset $F \subseteq R_c^s$ such that*

(1) *every element of R_c^s is isomorphic to an element of $X_c.F$;*
(2) *if $\tau, \tau' \in F$ and $\tau \neq \tau'$, then no element of $X_c.\tau$ is isomorphic to an element of $X_c.\tau'$;*
(3) *for any $\tau \in F$, there is a finite subgroup $X_\tau \subseteq X_c$ such that, for all $\chi, \chi' \in X_c$, we have: $\chi.\tau$ is isomorphic to $\chi'.\tau$ iff $\chi^{-1}\chi' \in X_\tau$; and*
(4) *if $\mathcal{H}$ is connected and algebraically simply connected, if every almost $\mathbb{Q}$-simple factor of $\mathcal{H}/\mathcal{N}$ is $\mathbb{R}$-isotropic, if $\tau \in F$, if $\chi \in X_c$, and if $K > 0$ is an integer, then: $\mathcal{H}(K) \subseteq \ker(\chi.\tau) \implies \mathcal{H}(K) \subseteq \ker(\tau)$.*

Proof: Let X denote the group of all one-dimensional representations of $\mathcal{H}(\mathbb{Z})$ whose restriction to $\mathcal{G}(\mathbb{Z})$ is trivial.

For each $\tau \in R_c$, let $K_\tau := \min\{K \in \mathbb{N} \mid \mathcal{H}(K) \subseteq \ker(\tau)\}$.

By Theorem 5.12, there are only finitely many ν-dimensional congruence representations $\eta_1, \ldots, \eta_s$ for $\mathcal{G}$.

Let i be an integer satisfying $1 \leq i \leq s$. Let $H := \mathcal{H}(\mathbb{Z})$. By Corollary 3.10, η_i gives rise to a finite set F_i of representations of H. Let E_i denote those elements $\tau \in F_i$ such that $X.\tau \cap R_c^s \neq \emptyset$. For each $\tau \in E_i$, choose $\tau' \in (X.\tau) \cap R_c^s$, such that

$$K_{\tau'} = \min\{K_\mu \mid \mu \in (X.\tau) \cap R_c^s\}.$$

Let $E_i' := \{\tau' \mid \tau \in E_i\}$.

Let $F := \bigcup_i E_i'$. Then (1), (2) and (3) follow from Corollary 3.10 and (4) follows from Lemma 5.6. ∎

COROLLARY 5.14: *Assume that $\mathcal{H}/\mathcal{N}$ is semisimple. Let $\mathcal{G}$ be a $\mathbb{Q}$-Levi factor of $\mathcal{H}$ and let $G := \mathcal{G}(\mathbb{Z})$. Assume that the Zariski closure of G is connected. Let*

A denote the Abelianization of $\mathcal{N}(\mathbb{Z})$. Let $\hat{A} := \mathrm{Hom}(A, \mathbb{T})$ be the dual group to A and assume that $\hat{A}^G$ is finite. Then, for every integer $\nu > 0$, $\mathcal{H}$ has only finitely many ν-dimensional congruence representations, up to isomorphism.

Proof: Every ν-dimensional congruence representation has finite image and is therefore completely reducible. So it suffices to show, for every integer $\nu > 0$, that there are finitely many *irreducible* ν-dimensional congruence representations, up to isomorphism. Fix an integer $\nu > 0$.

Let m', $\mathcal{H}'$, π, $\mathcal{G}'$ and $\mathcal{N}'$ be as in Lemma 5.1. By (3) of Lemma 5.1, the composition with π of any congruence representation for $\mathcal{H}$ is a congruence representation of $\mathcal{H}'$. By (1) and (2) of Lemma 5.1, $\pi(\mathcal{H}'(\mathbb{Z})) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$ is a subgroup of finite index in $\mathcal{H}(\mathbb{Z})$, so, by Lemma 3.6, each finite image representation of $\pi(\mathcal{H}'(\mathbb{Z}))$ has only finitely many extensions to $\mathcal{H}(\mathbb{Z})$, up to isomorphism. Therefore, replacing m by m', $\mathcal{H}$ by $\mathcal{H}'$, $\mathcal{G}$ by $\mathcal{G}'$ and $\mathcal{N}$ by $\mathcal{N}'$, we may assume that $\mathcal{H}(\mathbb{Z}) = \mathcal{G}(\mathbb{Z})\mathcal{N}(\mathbb{Z})$.

As in Theorem 5.13, let X_c denote the set of all one-dimensional congruence representations for $\mathcal{H}$ whose restriction to $\mathcal{G}(\mathbb{Z})$ is trivial. The restriction map $X_c \to \hat{A}$ is injective and has image $\hat{A}^G$. So, if $\hat{A}^G$ is finite, then X_c must be as well and the result now follows from (1) of Theorem 5.13.

6. Betti numbers

Throughout this section, all homology and cohomology groups are assumed to have coefficients in $\mathbb{C}$.

Let M be a connected topological space with the homotopy type of a finite CW complex. Let $\Gamma := \pi_1(M)$ and let $\tilde{M}$ denote the universal cover of M.

Fix an integer $m > 0$. Let $\mathcal{H}$ be a connected, algebraically simply connected $\mathbb{Q}$-subgroup of SL_m. Let $\mathcal{N}$ be the unipotent radical of $\mathcal{H}$ and let $\mathcal{G}$ be a reductive $\mathbb{Q}$-Levi factor. Then $\mathcal{G}$ is connected, semisimple and algebraically simply connected. Let $G := \mathcal{G}(\mathbb{Z})$, $N := \mathcal{N}(\mathbb{Z})$, $H := \mathcal{H}(\mathbb{Z})$.

Assume that every almost $\mathbb{Q}$-simple factor of $\mathcal{G}$ is $\mathbb{R}$-isotropic. Equivalently, assume that $\mathcal{G}(\mathbb{R})$ has no nontrivial compact normal subgroups. By [Borl, Theorem 1, p. 635], G is Zariski dense in $\mathcal{G}$. In particular, the Zariski closure in $\mathcal{G}$ of G is connected.

Let $\hat{H}$ be the set of isomorphy classes of irreducible finite dimensional complex representations of H. Let X denote the kernel of the restriction map

$\mathrm{Hom}(H, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$. Recall from §3 that X is the $\mathbb{C}$-points of a linear algebraic $\mathbb{Q}$-group. Let X^0 denote the identity component of X. As in §3, X^0 is the $\mathbb{C}$-points of a $\mathbb{Q}$-split torus.

Let V be any complex vector space. For any representation $\tau: H \to \mathrm{GL}(V)$, for any $\chi \in X$, we define a representation $\chi.\tau: H \to \mathrm{GL}(V)$ by $(\chi.\tau)(h) = \chi(h)\tau(h)$.

Let $\Phi: \Gamma \to H$ be a surjective homomorphism.

Fix an integer $K > 0$. Let $\mathrm{Rd}_K^H: \mathcal{H}(\mathbb{Z}) \to \mathcal{H}(\mathbb{Z}/K\mathbb{Z})$, $\mathrm{Rd}_K^G: \mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/K\mathbb{Z})$ and $\mathrm{Rd}_K^N: \mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}/K\mathbb{Z})$ denote reduction modulo K. Let

$$\mathcal{H}(K) := \ker(\mathrm{Rd}_K^H), \qquad \mathcal{G}(K) := \ker(\mathrm{Rd}_K^G), \qquad \mathcal{N}(K) := \ker(\mathrm{Rd}_K^N).$$

Let $H_K := \mathrm{Rd}_K^H(H)$. Let $M_K := \tilde{M} \times_\Gamma H_K$ be the covering associated to H_K via Φ. Let $\hat{H}_K := \{\tau \in \hat{H} \mid \mathcal{H}(K) \subseteq \ker(\tau)\}$. Let $X_K := \{\chi \in X \mid \mathcal{H}(K) \subseteq \ker(\chi)\}$. For each integer $\nu > 0$, let $\hat{H}_K^\nu := \{\tau \in \hat{H}_K \mid \dim(\tau) = \nu\}$.

Let A, B, C and D be (not necessarily commutative) rings. Let W be an (A, B)-bimodule, i.e., an additive Abelian group with left A-module structure and right B-module structure such that the left A-action commutes with the right B-action. Let Y be a (B, C)-bimodule and let Z be a (D, C)-bimodule. Define an (A, C)-bimodule structure on $W \otimes_B Y$ by $a.(w \otimes y).c = (a.w) \otimes (y.c)$. Define a (D, A)-bimodule structure on$\mathrm{Hom}_C(W \otimes_B Y, Z)$ by $(d.f.a)(w) = d.(f(a.w))$. Similarly, define a (D, B)-bimodule structure on $\mathrm{Hom}_C(Y, Z)$ by $(d.f.b)(y) = d.(f(b.y))$. Finally, define a (D, A)-bimodule structure on $\mathrm{Hom}_B(W, \mathrm{Hom}_C(Y, Z))$ by $(d.f.a)(w) = d.(f(a.w))$. Then the following is an exercise in homological algebra (see [Jac1, Proposition 3.8, p. 136]):

LEMMA 6.1: *In the category of (D, A)-bimodules,* $\mathrm{Hom}_C(W \otimes_B Y, Z)$ *is isomorphic to* $\mathrm{Hom}_B(W, \mathrm{Hom}_C(Y, Z))$.

COROLLARY 6.1: *For all integers $q, K > 0$, we have*

$$\beta^q(M_K) := \sum_{\tau \in \hat{H}_K} \dim(\tau) \cdot \beta_\Gamma^q(\tilde{M}, \tau).$$

Proof: Fix integers $q, K > 0$.

Let $A := \mathbb{C}$. Let $B := \mathbb{C}[\Gamma]$, the group ring of Γ. Let $C := \mathbb{C}$ and $D := \mathbb{C}$. Let $W := C_*(M)$ be the module of chains on M with coefficients in $\mathbb{C}$. Let $Y := \mathbb{C}[H_K]$ be the group ring of H_K. Let B act on Y via $\gamma.y = \mathrm{Rd}_K^H(\Phi(\gamma))y$. Let $Z := \mathbb{C}$.

Fix an integer $K > 0$. Let $\mathcal{S}(M)$ and $\mathcal{S}(\tilde{M})$ denote, respectively, the set of parametric simplices in M and $\tilde{M}$. Then $C_*(M) = \mathbb{C}[\mathcal{S}(M)]$ and $C_*(\tilde{M}) = \mathbb{C}[\mathcal{S}(\tilde{M})]$. Let

$$\sigma \mapsto \sigma(0) \colon \mathcal{S}(M) \to M$$

be the footpoint map. By simplex lifting, $\mathcal{S}(\tilde{M})$ is isomorphic as a Γ-set with the fibered product $\mathcal{S}(M) \times_M \tilde{M}$. Let $\mathcal{S}(\tilde{M} \times_\Gamma H_K)$ denote the set of parametric simplices in $M \times_\Gamma H_K$. Simplex lifting also shows that, as H_K-spaces,

$$\mathcal{S}(\tilde{M} \times_\Gamma H_K) \simeq \mathcal{S}(M) \times_M (\tilde{M} \times_\Gamma H_K) \simeq (\mathcal{S}(M) \times_M \tilde{M}) \times_\Gamma H_K \simeq \mathcal{S}(\tilde{M}) \times_\Gamma H_K.$$

Therefore we obtain an ismorphism of chain complexes:

$$\mathbb{C}[\mathcal{S}(\tilde{M} \times_\Gamma H_K)] \simeq \mathbb{C}[\mathcal{S}(\tilde{M})] \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[H_K],$$

or $C_*(\tilde{M} \times_\Gamma H_K) \simeq C_*(\tilde{M}) \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[H_K]$.

In the category of (A, C)-bimodules

$$W \otimes_B Y = C_*(\tilde{M}) \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[H_K] \cong C_*(\tilde{M} \times_\Gamma H_K) = C_*(M_K).$$

Since H_K is finite, it follows that $\mathbb{C}[H_K]$ admits a Γ-invariant Hermitian inner product. Thus, in the category of (D, B)-bimodules,

$$\mathrm{Hom}_{\mathbb{C}}(Y, Z) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[H_K], \mathbb{C}) \cong \mathbb{C}[H_K].$$

By Lemma 6.1, we conclude

$$\mathrm{Hom}_{\mathbb{C}}(C_*(M_K), \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}[\Gamma]}(C_*(\tilde{M}), \mathbb{C}[H_K]).$$

This isomorphism is an isomorphism of cochain complexes. Taking cohomology, then taking dimension, we have, for all integers $q > 0$: $\beta^q(M_K) = \beta^q_\Gamma(\tilde{M}, \mathbb{C}[H_K])$. The decomposition of the regular representation (for finite groups) implies that, in the category of $\mathbb{C}[\Gamma]$-modules,

$$\mathbb{C}[H_K] \cong \bigoplus_{\tau \in \hat{H}_K} \dim(\tau) \cdot \tau.$$

The result follows. ∎

Definition 6.3: For all integers $K, \nu, q > 0$, define

$$\beta_K^{q,\nu} := \nu \cdot \sum_{\tau \in \hat{H}_K^\nu} \beta_\Gamma^q(\tilde{M}, \tau).$$

By Corollary 6.2, for all integers $q, K > 0$, we have $\beta^q(M_K) = \sum_{\nu=1}^{\infty} \beta_K^{q,\nu}$.

Definition 6.4: Let $P: \mathbb{N} \to \mathbb{R}$ be an $\mathbb{R}$-valued function defined on the positive integers. We say that P is **polynomial periodic** if there exist an integer $s > 0$ and polynomial functions $Q_1, \ldots, Q_s: \mathbb{R} \to \mathbb{R}$ such that: for all integers $a \geq 0$, for all integers r satisfying $1 \leq r \leq s$, we have $P(as + r) = Q_r(a)$.

We aim (Theorem 6.7) to prove that if $H = GN$, and if $\nu, q > 0$ are integers, then the function $K \mapsto \beta_K^{q,\nu}$ is polynomial periodic.

The next lemma is the point of [SarAd1, equation (2.14)]. The work in [SarAd1] is required for the more general framework of cohomological representations. We do not require that level of generality here and are therefore able to make the following proof (suggested by T. Mrowka) which does not use combinatorial Hodge theory.

LEMMA 6.5: *Let V be a complex vector space. Let $\tau: H \to \mathrm{GL}(V)$ be a finite dimensional complex representation. Let $q > 0$ be an integer. Then the map*

$$\chi \mapsto \beta_\Gamma^q(\tilde{M}, \chi.\tau): X^0 \to \mathbb{N} \cup \{0\}$$

is Zariski upper semicontinuous.

Proof: Let $\nu := \dim V$. We assume without loss of generality that $V = \mathbb{C}^\nu$.

Recall that we are assuming that M has the homotopy type of a finite CW complex. Replacing M by a finite CW complex that is homotopy equivalent to M, we may assume that M *is* a finite CW complex. For each integer $r \geq 0$, let M^r denote the r-skeleton of M. For all integers $r < 0$, let $M^r := \emptyset$.

Let $c_1, \ldots, c_t$ be a $\mathbb{C}$-basis for the complex vector space

$$C_q(M) := H_q(M^q, M^{q-1}).$$

Let $b_1, \ldots, b_s$ be a $\mathbb{C}$-basis for $C_{q+1}(M) := H_{q+1}(M^{q+1}, M^q)$. Let $\tilde{b}_1, \ldots, \tilde{b}_s$, $\tilde{c}_1, \ldots, \tilde{c}_t$ denote lifts to $\tilde{M}$ of $b_1, \ldots, b_s, c_1, \ldots, c_t$.

Define a matrix $[z_{ij}] \in \mathbb{C}^{s\times t}$ by: for all $i = 1, \dots, s$, $\partial b_i = \sum_{j=1}^{t} z_{ij} c_j$. Define a matrix $[\gamma_{ij}] \in \Gamma^{s\times t}$ by: for all $i = 1, \dots, s$, $\partial \tilde{b}_i = \sum_{j=1}^{t} z_{ij}\gamma_{ij}\tilde{c}_j$. Recall that $\Phi\colon \Gamma \to H$ is a surjective homomorphism. For all i, j, define $h_{ij} := \Phi(\gamma_{ij}) \in H$.

Fix $\chi \in X^0$. An element of $C^q_\Gamma(\tilde{M}, \chi.\tau)$ is determined by its values on $\tilde{c}_1, \dots, \tilde{c}_t$, defining an isomorphism between $C^q_\Gamma(\tilde{M}, \chi.\tau)$ and $\mathbb{C}^{(t\nu)\times 1}$. Similarly, evaluation on $\tilde{b}_1, \dots, \tilde{b}_s$ defines an isomorphism between $C^{q+1}_\Gamma(\tilde{M}, \chi.\tau)$ and $\mathbb{C}^{(s\nu)\times 1}$. Then the coboundary operator $\delta^q_\chi\colon C^q_\Gamma(\tilde{M}, \chi.\tau) \to C^{q+1}_\Gamma(\tilde{M}, \chi.\tau)$ becomes identified with a matrix $\epsilon_\chi \in \mathbb{C}^{(s\nu)\times(t\nu)}$. The matrix ϵ_χ breaks up into st blocks, each in $\mathbb{C}^{\nu\times\nu}$. Let I denote the $\nu \times \nu$ identity matrix. For each i, j, the (i, j)-block of ϵ_χ is $[z_{ij} I] \cdot \left[(\chi.\tau)(\Phi(\gamma_{ij}))\right] = [z_{ij} I] \cdot \left[\tau(h_{ij})\right] \cdot \left[\chi(h_{ij}) I\right]$.

For each $h \in H$, the map $\chi \mapsto \chi(h)\colon X^0 \to \mathbb{C}$ is a regular function on X^0. It therefore follows from the above expression for the (i, j)-block that: for all $\alpha = 1, \dots, s\nu$, for all $\beta = 1, \dots, t\nu$, the map $X^0 \to \mathbb{C}$ which sends χ to the (α, β)-entry of ϵ_χ is a regular function. Consequently, $\chi \mapsto \dim(\ker \epsilon_\chi)\colon X^0 \to \mathbb{N} \cup \{0\}$ is Zariski upper semicontinuous. For all $\chi \in X^0$, by construction of ϵ_χ, we have $\dim(\ker \epsilon_\chi) = \dim(\ker \delta^q_\chi)$.

For each $\chi \in X^0$, let $\delta^{q-1}_\chi\colon C^{q-1}_\Gamma(\tilde{M}, \chi.\tau) \to C^q_\Gamma(\tilde{M}, \chi.\tau)$ be the coboundary operator. An analogous argument shows that $\chi \mapsto \dim(\ker \delta^{q-1}_\chi)\colon X^0 \to \mathbb{N} \cup \{0\}$ is Zariski upper semicontinuous, so $\chi \mapsto \dim(\operatorname{im} \delta^{q-1}_\chi)\colon X^0 \to \mathbb{N} \cup \{0\}$ is Zariski lower semicontinuous.

For all $\chi \in X^0$, we have $\beta^q_\Gamma(\tilde{M}, \chi.\tau) = \dim(\ker \delta^q_\chi) - \dim(\operatorname{im} \delta^{q-1}_\chi)$, so $\chi \mapsto \beta^q_\Gamma(\tilde{M}, \chi.\tau)$ is Zariski upper semicontinous, as desired. ∎

LEMMA 6.6: *Assume that $H = GN$. Let V be a finite dimensional complex vector space. Let $\tau\colon H \to \mathrm{GL}(V)$ be a representation of H. Fix an integer $q > 0$. Let $\phi \in X$. For all integers $K > 0$, define $Y_K := (\phi \cdot X^0) \cap X_K$ and*

$$f(K) := \sum_{\chi \in Y_K} \beta^q_\Gamma(\tilde{M}, \chi.\tau).$$

Then $f\colon \mathbb{N} \to \mathbb{Z}$ is polynomial periodic.

Proof: Let A denote the Abelianization of N. Let pr: $N \to A$ denote the projection map. Choose an integer $n > 0$ such that $\phi^n \in X^0$. The group X^0 is a torus and is therefore divisible; choose $\psi \in X^0$ such that $\psi^n = \phi^n$. Replacing ϕ by $\phi\psi^{-1}$, we may assume that ϕ has finite order in the group X.

For all integers $K > 0$, let $Z_K := X^0 \cap (\phi^{-1} \cdot X_K) = \phi^{-1} \cdot Y_K$; then

$$f(K) = \sum_{\chi \in Z_K} \beta^q_\Gamma(\tilde{M}, (\phi\chi).\tau).$$

For all integers $\beta_0 > 0$, let $X^{\beta_0} := \{\chi \in X^0 \mid \beta^q_\Gamma(\tilde{M}, (\phi\chi).\tau) \geq \beta_0\}$. Then, for all integers $K > 0$, we have

$$f(K) = \sum_{\beta_0 \in \mathbb{N}} |Z_K \cap X^{\beta_0}|,$$

where $|\cdot|$ denotes cardinality.

Fix an integer $\beta_0 > 0$. We wish to show that $K \mapsto |Z_K \cap X^{\beta_0}|: \mathbb{N} \to \mathbb{Z}$ is polynomial periodic.

Let X_1 denote the elements $\chi \in X$ such that $\chi(H) \subseteq \mathbb{T}$. The connected component X_1^0 of the identity in X_1 is a compact torus. Since $H = GN$, the definition of X shows that there is a natural identification of X_1 with $\mathrm{Hom}_G(N, \mathbb{T})$ coming from the restriction map $X_1 \subseteq \mathrm{Hom}(H, \mathbb{T}) \to \mathrm{Hom}(N, \mathbb{T})$. Note that $\mathrm{Hom}(N, \mathbb{T}) = \hat{A}$ is the dual group to A. So we have identified X_1^0 with a closed connected subgroup of $\hat{A}$.

Let $\mathrm{tor}X_1^0$ denote the set of elements of X_1^0 of finite order. Any element of X with finite image has finite order. Since ϕ has finite order and since every element of every X_K has finite image, we conclude: for all integers $K > 0$, we have $Z_K \subseteq \mathrm{tor}X_1^0$.

By Lemma 6.5, $X_1^{\beta_0}$ is Zariski closed, so, by [SarAd1, Proposition 1.6], there exist a finite number of rational planes $\pi_1, \ldots, \pi_l$ contained in X_1^0 such that

$$(\mathrm{tor}X_1^0) \cap X^{\beta_0} = (\mathrm{tor}X_1^0) \cap (\pi_1 \cup \cdots \cup \pi_l).$$

Therefore, for all integers $K > 0$, we have

$$Z_K \cap X^{\beta_0} = Z_K \cap (\pi_1 \cup \cdots \cup \pi_l) = (\phi^{-1} \cdot X_K) \cap (\pi_1 \cup \cdots \cup \pi_l).$$

By inclusion-exclusion, it now suffices to show: for all rational planes $\pi \subseteq X_1^0$, the function

$$K \mapsto |(\phi^{-1} \cdot X_K) \cap \pi|: \mathbb{N} \to \mathbb{Z}$$

is polynomial periodic.

The character ϕ factors to a character ϕ_0 on A. Because of the identification of X_0^1 with a closed subgroup of $\hat{A}$, the rational plane π becomes identified with a rational plane π_0 in $\hat{A}$.

For all integers $K > 0$, let $A(K)$ denote the image of $\mathcal{N}(K)$ in A; then,

$$\begin{aligned}(\phi^{-1} \cdot X_K) \cap \pi &= \{\chi \in \pi : (\phi\chi)|\mathrm{pr}^{-1}(A(K)) = 1\} \\ &= \{\chi \in \pi_0 : (\phi_0\chi)|A(K) = 1\}.\end{aligned}$$

The result now follows from [SarAd1, Theorem 10.1] (where $\mathcal{N}$ is denoted by H, A is denoted by Γ, ϕ_0 is denoted by χ_0 and π_0 is denoted by π). ∎

I am not sure if the assumption that $H = GN$ is necessary in Lemma 6.6 above or Theorem 6.7 below. Theorem 6.7 is the main result of this section.

THEOREM 6.7: *Assume that $H = GN$. Fix integers $q, \nu > 0$. Then $K \mapsto \beta_K^{q,\nu} \colon \mathbb{N} \to \mathbb{Z}$ is polynomial periodic.*

Proof: Let $V := \mathbb{C}^\nu$. Fix F as in Lemma 5.13. For all $\tau \in F$, let X_τ be as in (3) of Lemma 5.13. For any integer $K > 0$, define

$$F_K := \{\tau \in F \,|\, \mathcal{H}(K) \subseteq \ker(\tau)\},$$
$$X_K := \{\chi \in X \,|\, \mathcal{H}(K) \subseteq \ker(\chi)\}.$$

By (4) of Lemma 5.13, if $K > 0$ is an integer, if $\tau \in F \backslash F_K$ and if $\chi \in X$, then $\chi.\tau \notin H_K^\nu$. By (3) of Lemma 5.13, for all integers $K > 0$, for all $\tau \in F_K$, we have

$$\chi \in X_\tau \iff \chi.\tau \simeq \tau;$$

as $\chi.\tau \simeq \tau \implies \chi \in X_K$, we see that $X_\tau \subseteq X_K$.

From these observations and from Definition 6.3, we have

$$\beta_K^{q,\nu} = \nu \cdot \sum_{\tau \in \hat{H}_K^\nu} \beta_\Gamma^q(\tilde{M}, \tau) = \sum_{\tau \in F_K} \frac{\nu}{|X_\tau|} \sum_{\chi \in X_K} \beta_\Gamma^q(\tilde{M}, \chi.\tau).$$

Let T be a set of coset representatives for X^0 in X. For any integer $K > 0$, for any $\phi \in T$, let $Y_K^\phi := (\phi \cdot X^0) \cap X_K$. Then, for any integer $K > 0$, we have

$$\beta_K^{q,\nu} = \sum_{\tau \in F_K} \frac{\nu}{|X_\tau|} \sum_{\phi \in T} \sum_{\chi \in Y_K^\phi} \beta_\Gamma^q(\tilde{M}, \chi.\tau).$$

By Corollary 5.4, $K \mapsto F_K$ is periodic. By Lemma 6.6, for all $\tau \in F$, for all $\phi \in T$,

$$K \mapsto \sum_{\chi \in Y_K^\phi} \beta_\Gamma^q(\tilde{M}, \chi.\tau)$$

is polynomial periodic. The result follows. ∎

We also have the following periodicity result. It does not require the assumption that $H = GN$.

THEOREM 6.8: *Let A denote the Abelianization of N, let $\hat{A} := \mathrm{Hom}(A, \mathbb{T})$ denote the dual group to A. Assume that $\hat{A}^G$ is finite. Fix integers $q, \nu > 0$. Then $K \mapsto \beta_K^{q,\nu}\colon \mathbb{N} \to \mathbb{Z}$ is periodic.*

Proof: Let $\hat{H}_c^\nu$ denote the set of all isomorphism classes of ν-dimensional congruence representations for $\mathcal{H}$. By Corollary 5.14, $\hat{H}_c^\nu$ is finite. For all integers $K > 0$, we have $\hat{H}_K^\nu := \{\tau \in \hat{H}_c^\nu \,|\, \mathcal{H}(K) \subseteq \ker(\tau)\}$. By Corollary 5.4, $K \mapsto \hat{H}_K^\nu$ is periodic; by Definition 6.3, $\beta_K^{q,\nu}$ is periodic as well. ∎

7. Appendix 1: A group with no $\mathbb{Z}$-splitting Levi factor

In this appendix, our goal is to construct an algebraic $\mathbb{Q}$-subgroup of some SL_m such that, for any reductive $\mathbb{Q}$-Levi factor, there is a $\mathbb{Z}$-point of the group which cannot be written as the product of a $\mathbb{Z}$-point in the Levi factor and a $\mathbb{Z}$-point in the unipotent radical. The example we construct is connected and algebraically simply connected (see Example 7.2).

On the other side of the coin, we have Lemma 5.1 which tells us: given a linear algebraic $\mathbb{Q}$-group and a $\mathbb{Q}$-Levi factor, there is an embedding into some $\mathrm{SL}_{m'}$ such that every $\mathbb{Z}$-point of the group may be written as the product of a $\mathbb{Z}$-point of the Levi factor and a $\mathbb{Z}$-point of the unipotent radical.

For each $X, Y \in \mathrm{SL}_2$, let $R_{XY}^0 \in \mathrm{SL}_4$ be the matrix with upper left (2×2)-block equal to X, lower right (2×2)-block equal to Y, and all other entries zero. Let $R_{XY} \in \mathrm{SL}_5$ denote the matrix with upper left (4×4)-block equal to R_{XY}^0, lower right (1×1)-block equal to 1, and all other entries zero. Let $\mathbb{A}^{2\times 1}$ denote the affine variety of all 2×1 column matrices. For all $v \in \mathbb{A}^{2\times 1}$: let v_1, v_2 denote the $(1,1)$ and $(2,1)$ entries of v, respectively, then let $U_v \in \mathrm{SL}_5$ be the matrix with 1s on the diagonal, with v_1 and v_2 in the $(3,5)$ and $(4,5)$ entries and with all other entries zero. Let $\mathcal{G} := \{R_{XY}U_v |\ X, Y \in \mathrm{SL}_2, v \in \mathbb{A}^{2\times 1}\}$. Then $\mathcal{G}$ is a $\mathbb{Q}$-subgroup of SL_5.

Let $\Gamma \subseteq \mathbb{Z}^{4\times 1}$ be the set of all integer column matrices such that: the sum of the $(2,1)$ entry and the $(4,1)$ entry is even. For each $\gamma \in \Gamma$, let $\gamma' \in \mathbb{Z}^{5\times 1}$ be the matrix whose top four entries are the entries of γ and whose $(5,1)$ entry is zero. Let $\Gamma' := \{\gamma' \,|\, \gamma \in \Gamma\}$. Let $\lambda \in \mathbb{Q}^{5\times 1}$ be the column matrix whose entries are $(0, 1/2, 0, 0, 1)$, from top to bottom. Let $\Lambda := \Gamma' + \mathbb{Z}\lambda$. Let

$$A := \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \qquad w := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{Z}^{2\times 1}.$$

Let $I \in \mathrm{SL}_2(\mathbb{Z})$ be the identity matrix. Let $C := R_{AI}$.

Fix $v \in \mathbb{Q}^{2\times 1}$. Let $\lambda_v := U_v\lambda$ and let $\Lambda_v := U_v\Lambda$. Since $U_v\Gamma' = \Gamma'$, we find that $\Lambda_v := \Gamma' + \mathbb{Z}\lambda_v$.

LEMMA 7.1: *For all $v \in \mathbb{Q}^{2\times 1}$, we have*
(i) *$C\Lambda_v \neq \Lambda_v$; and*
(ii) *$(U_wC)\Lambda_v = \Lambda_v$.*

Proof: We will use $(a,b,c,d,e)^t$ to denote the 5×1 column matrix whose entries are (a,b,c,d,e) from top to bottom. Fix $v \in \mathbb{Q}^{2\times 1}$.

Proof of (i): For all $m \in \mathbb{Z}$, the $(5,1)$ entry of $C\lambda_v - m\lambda_v$ is $1-m$. So, for all $m \in \mathbb{Z}\backslash 1$, we have $C\lambda_v - m\lambda_v \notin \Gamma'$. But also, we have $C\lambda_v - \lambda_v = (-1,1,0,0,1)^t \notin \Gamma'$. So $C\lambda_v \notin \Gamma' + \mathbb{Z}\lambda_v = \Lambda_v$. On the other hand, $C\lambda_v \in C\Lambda_v$, so $C\Lambda_v \neq \Lambda_v$.

Proof of (ii): A $\mathbb{Z}$-basis for Γ' is given by

$$(1,0,0,0,0)^t,\ (0,0,1,0,0)^t,\ (0,1,0,1,0)^t,\ (0,0,0,2,0)^t,$$

so, by definition of C, we find that

$$(-1,2,0,0,0)^t,\ (0,0,1,0,0)^t,\ (-2,3,0,1,0)^t,\ (0,0,0,2,0)^t$$

is a $\mathbb{Z}$-basis for $C\Gamma'$. Thus $C\Gamma' \subseteq \Gamma'$, and, by equality of covolume in $(\mathbb{R}^4 \times \{0\})^t$, we have $C\Gamma' = \Gamma'$. But $U_w\Gamma' = \Gamma'$, so $(U_wC)\Gamma' = \Gamma'$.

By matrix multiplication, one verifies that

$$\gamma' := (U_wC)\lambda_v - \lambda_v = (-1,1,0,1,0)^t \in \Gamma'.$$

Now

$$(U_wC)\Lambda_v = (U_wC)\Gamma' + \mathbb{Z}[(U_wC)\lambda_v],$$

so we have $(U_wC)\Lambda_v = \Gamma' + \mathbb{Z}(\gamma' + \lambda_v)$. But $\gamma' \in \Gamma'$, so $(U_wC)\Lambda_v = \Gamma' + \mathbb{Z}\lambda_v = \Lambda_v$. ∎

Example 7.2: Let $M \in \mathrm{GL}_5(\mathbb{Q})$ satisfy $M\Lambda = \mathbb{Z}^{5\times 1}$. Let $\mathcal{H} := M\mathcal{G}M^{-1}$. Let $\mathcal{U}$ denote the unipotent radical of $\mathcal{H}$ Then, for every $\mathbb{Q}$-Levi factor $\mathcal{L}$ of $\mathcal{H}$, we have $\mathcal{U}(\mathbb{Z})\mathcal{L}(\mathbb{Z}) \neq \mathcal{H}(\mathbb{Z})$.

Proof: For all $v \in \mathbb{Q}^{2\times 1}$, let

$$\mathcal{L}_v := MU_v^{-1}\{R_{XY} \mid X, Y \in \mathrm{SL}_2\}U_vM^{-1}.$$

Then $\{\mathcal{L}_v |\, v \in \mathbb{Q}^{2\times 1}\}$ is the set of all reductive $\mathbb{Q}$-Levi factors of $\mathcal{H}$.

Fix $v \in \mathbb{Q}^{2\times 1}$. We wish to show that $\mathcal{U}(\mathbb{Z})\mathcal{L}_v(\mathbb{Z}) \neq \mathcal{H}(\mathbb{Z})$.

Let $H := MU_v^{-1}(U_w C)U_v M^{-1}$. There exist unique elements $U \in \mathcal{U}(\mathbb{Q})$, $L \in \mathcal{L}_v(\mathbb{Q})$ such that $H = UL$. We wish to show that $H \in \mathcal{H}(\mathbb{Z})$ and that $L \notin \mathcal{L}_v(\mathbb{Z})$. That is, we wish to show that $H\mathbb{Z}^{5\times 1} = \mathbb{Z}^{5\times 1}$, while $L\mathbb{Z}^{5\times 1} \neq \mathbb{Z}^{5\times 1}$. By uniqueness, we must have

$$U = MU_v^{-1}(U_w)U_v M^{-1}, \qquad L = MU_v^{-1}(C)U_v M^{-1}.$$

By Lemma 7.1, we have

$$\begin{aligned} H\mathbb{Z}^{5\times 1} &= (MU_v^{-1})(U_w C)\Lambda_v = (MU_v^{-1})\Lambda_v = \mathbb{Z}^{5\times 1}, \\ L\mathbb{Z}^{5\times 1} &= (MU_v^{-1})(C)\Lambda_v \neq (MU_v^{-1})\Lambda_v = \mathbb{Z}^{5\times 1}. \qquad \blacksquare \end{aligned}$$

8. Appendix 2: Lack of polynomial periodicity in algebraic tori

In this appendix, we will show that polynomial periodicity of Betti numbers (Theorem 6.7) does not hold without the assumption that the reductive Levi factor of the algebraic group is semisimple. See the remarks at the end of Appendix 2. I would like to thank Z. Rudnick for showing me a variant of the argument presented below.

Fix a positive integer d such that $d \notin \{n^2 \,|\, n \in \mathbb{Z}\}$. Define

$$\mathcal{G}_0 := \left\{ \begin{bmatrix} x & y \\ dy & x \end{bmatrix} : x^2 - dy^2 = 1 \right\}.$$

Let $I \in \mathcal{G}_0(\mathbb{Z})$ denote the identity matrix and let $-I \in \mathcal{G}_0(\mathbb{Z})$ denote its negative. For all $g \in \mathcal{G}_0$, let $-g := (-I)g$.

For all integers $N > 0$, let $\mathrm{Rd}_N\colon \mathcal{G}_0(\mathbb{Z}) \to \mathcal{G}_0(\mathbb{Z}/N\mathbb{Z})$ denote reduction mod N, let $\mathcal{G}_0(N) := \ker(\mathrm{Rd}_N)$ and let $G_N := \mathrm{Rd}_N(\mathcal{G}_0(\mathbb{Z}))$. We intend to prove that $N \mapsto |G_N|$ is not polynomial periodic (see Proposition 8.7). In the remarks following the proof of Proposition 8.7, we will explain why the results in §6 can fail if we do not make the assumption of semisimplicity of the reductive Levi factor of the algebraic group used to define the congruence tower.

Let $\mathrm{Ext}_{\mathbb{Z}}^{\mathbb{Q}}$ denote the extension of scalars functor from schemes over $\mathrm{Spec}(\mathbb{Z})$ to schemes over $\mathrm{Spec}(\mathbb{Q})$. A **$\mathbb{Z}$-structure** on a $\mathbb{Q}$-variety $\mathcal{V}$ is a scheme $\mathcal{V}_*$ over $\mathrm{Spec}(\mathbb{Z})$, together with an isomorphism $\mathrm{Ext}_{\mathbb{Z}}^{\mathbb{Q}}(\mathcal{V}_*) \to \mathcal{V}$. A **$\mathbb{Z}$-variety** is a $\mathbb{Q}$-variety with $\mathbb{Z}$-structure. Note that $\mathcal{G}_0$ is a $\mathbb{Z}$-variety, where the $\mathbb{Z}$-structure comes from the embedding of $\mathcal{G}_0$ in the affine space of 2×2 matrices.

If $\mathcal{V}$ is a $\mathbb{Z}$-variety and if A is a $\mathbb{Z}$-algebra, then we define $\mathcal{V}(A)$ to be the A-points of the Z-structure underlying $\mathcal{V}$. Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For any $\mathbb{Z}$-variety $\mathcal{V}$, we define $|\mathcal{V}| := |\mathcal{V}(\bar{\mathbb{Q}})|$; this number is usually infinite. We say that $\mathcal{V}$ is **nonempty** if $\mathcal{V}(\bar{\mathbb{Q}})$ is nonempty.

LEMMA 8.1: *Let $\mathcal{V}$ be a nonempty $\mathbb{Z}$-variety. Then, for infinitely many primes p, we have $\mathcal{V}(\mathbb{Z}/p\mathbb{Z}) \neq \emptyset$.*

Proof: We may assume that $\mathcal{V}$ is an affine $\mathbb{Q}$-variety with $\mathbb{Z}$-structure. We may therefore assume, for some integer d, that $\mathcal{V}$ is a $\mathbb{Q}$-closed subvariety of affine d-space with its $\mathbb{Z}$-structure inherited from the affine space.

By the Nullstellensatz, there is a numberfield K such that $\mathcal{V}(K) \neq \emptyset$. We may assume that K is Galois over $\mathbb{Q}$. Let O_K denote the ring of integers in K. Choose $\alpha \in (O_K)^d$, $b \in O_K$ such that $\alpha/b \in \mathcal{V}(K)$.

Let $\mathcal{A}$ denote the set of rational primes which split completely in O_K. Let $\mathcal{B}$ denote the set of all rational primes p such that, for every prime P in O_K lying over p, we have $b \in P$. By the Tchebotarev density theorem [La1, Theorem VIII.10, p. 169], we find that $\mathcal{A}$ is infinite. On the other hand, $\mathcal{B}$ is finite. Let $p \in \mathcal{A}\backslash\mathcal{B}$; we wish to show that $\mathcal{V}(\mathbb{Z}/p\mathbb{Z}) \neq \emptyset$.

Choose a prime P in O_K lying over p such that $b \notin P$. Since p splits completely in O_K, it follows that the residue degree $[O_K/P\colon \mathbb{Z}/p\mathbb{Z}] = 1$. So, as $\mathbb{Z}$-algebras, O_K/P is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It therefore suffices to show that $\mathcal{V}(O_K/P) \neq \emptyset$.

However, the image of b in O_K/P is nonzero, hence invertible, so the image of α/b in $(O_K/P)^d$ is defined and is an element of $\mathcal{V}(O_K/P)$. ∎

COROLLARY 8.2: *Let $m > 0$ be an integer, let $\mathcal{V}_1, \dots, \mathcal{V}_m$ be $\mathbb{Z}$-varieties. Let $n_1, \dots, n_m$ be positive integers. Assume, for all $i \in \{1, \dots, m\}$, that $|\mathcal{V}_i| \geq n_i$. Then, for infinitely many rational primes p, for all $i \in \{1, \dots, m\}$, we have $|\mathcal{V}_i(\mathbb{Z}/p\mathbb{Z})| \geq n_i$.*

Proof: For all $i \in \{1, \dots, m\}$, let $\mathcal{V}_i'$ be the open subvariety of $\mathcal{V}_i^{n_i}$ consisting of all points with n_i distinct coordinates; give $\mathcal{V}_i'$ the $\mathbb{Z}$-structure inherited from the product $\mathbb{Z}$-structure on $\mathcal{V}_i^{n_i}$. Let $\mathcal{V} := \mathcal{V}_1' \times \cdots \times \mathcal{V}_m'$. Then $\mathcal{V} \neq \emptyset$ and, applying Lemma 8.1 to $\mathcal{V}$ gives the desired result. ∎

LEMMA 8.3: *There exists $g_0 \in \mathcal{G}_0(\mathbb{Z})$ such that g_0 has infinite order and such that $\mathcal{G}_0(\mathbb{Z}) = \{g_0^k \mid k \in \mathbb{Z}\} \cup \{-g_0^k \mid k \in \mathbb{Z}\}$.*

Proof: Let $S := \{(x,y) \in \mathbb{R}^2 \mid x^2 - dy^2 = 1\}$ and let $S' := \{(u,v) \in \mathbb{R}^2 \mid uv = 1\}$. Let $\mathbb{R}^*$ denote the multiplicative group $\mathbb{R}\backslash\{0\}$.

Let $\phi\colon S \to S'$ be the homeomorphism defined by $\phi(x,y) = (x+y\sqrt{d}, x-y\sqrt{d})$. Let $P\colon S' \to \mathbb{R}^*$ be the homeomorphism defined by $P(u,v) = u$. Since P and ϕ are homeomorphisms, it follows that $P(\phi(S \cap \mathbb{Z}^2))$ is a discrete subset of $\mathbb{R}^*$.

Let $\psi\colon \mathcal{G}_0(\mathbb{Z}) \to \mathbb{R}^*$ be the injective homomorphism defined by

$$\psi\left(\begin{bmatrix} x & y \\ dy & x \end{bmatrix}\right) = x + y\sqrt{d}.$$

Let $\Gamma := \psi(\mathcal{G}_0(\mathbb{Z}))$. Since $\Gamma = P(\phi(S \cap \mathbb{Z}^2))$, it follows that Γ is a discrete subgroup of $\mathbb{R}^*$. Furthermore, for all $\gamma \in \Gamma$, we have $-\gamma \in \Gamma$. Therefore, there exists $\gamma_0 \in \Gamma$ such that $\Gamma = \{\gamma_0^k \mid k \in \mathbb{Z}\} \cup \{-\gamma_0^k \mid k \in \mathbb{Z}\}$. Let $g_0 := \psi^{-1}(\gamma_0)$. Then $\mathcal{G}_0(\mathbb{Z}) = \{\pm g_0^k \mid k \in \mathbb{Z}\}$.

It remains to show that g_0 has infinite order. Since $\mathcal{G}_0$ has no characters defined over $\mathbb{Q}$, it is a consequence of [Bor2, Proposition 8.5, p. 55] (with G replaced by $\mathcal{G}_0$ and G' replaced by the trivial group) that $\mathcal{G}_0(\mathbb{R})/\mathcal{G}_0(\mathbb{Z})$ is compact. As $\mathcal{G}_0(\mathbb{R})$ is noncompact, it follows that $\mathcal{G}_0(\mathbb{Z})$ is infinite. Since $\mathcal{G}_0(\mathbb{Z}) = \{\pm g_0^k \mid k \in \mathbb{Z}\}$, we conclude that g_0 must have infinite order. ∎

LEMMA 8.4: *Let p be a rational prime and assume that d is a square mod p. Then $\mathcal{G}_0(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$.*

Proof: Let $S := \{(u,v) \in (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z}) \mid uv = 1\}$. Then S becomes a group under component-by-component multiplication. Choose $c \in \mathbb{Z}/p\mathbb{Z}$ such that $c^2 \equiv d \bmod p$. Then

$$\begin{bmatrix} x & y \\ dy & x \end{bmatrix} \mapsto (x+cy, x-cy)\colon \mathcal{G}_0(\mathbb{Z}/p\mathbb{Z}) \to S$$

is an isomorphism and $(u,v) \mapsto u\colon S \to (\mathbb{Z}/p\mathbb{Z})^*$ is another isomorphism. ∎

LEMMA 8.5: *Let $\epsilon > 0$. Then there is a rational prime p such that $|G_p| < \epsilon p$.*

Proof: Let l be a positive integer such that $1 \leq l\epsilon$. It suffices to show that $l|G_p| < p$.

Let g_0 be as in Lemma 8.3. Let $m := 4$. Let $\mathcal{V}_1$ be the collection of square roots of d in the multiplicative group. Let $\mathcal{V}_2$ be the collection of lth roots of unity in the multiplicative group. Let $\mathcal{V}_3$ be the collection of lth roots of g_0 in $\mathcal{G}_0$. Let $\mathcal{V}_4$ be the collection of lth roots of $-I$ in $\mathcal{G}_0$. Let $n_1 := 1$, $n_2 := l$, $n_3 := 1$, $n_4 := 1$.

By Corollary 8.2, choose a rational prime p such that, for all $i \in \{1, \ldots, 4\}$, we have $|\mathcal{V}_i(\mathbb{Z}/p\mathbb{Z})| \geq n_i$.

Let $H := \mathcal{G}_0(\mathbb{Z}/p\mathbb{Z})$. Since $\mathcal{V}_1(\mathbb{Z}/p\mathbb{Z})$ is nonempty, it follows that d is a square mod p; by Lemma 8.4, H is isomorphic to the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$. Since $|\mathcal{V}_2(\mathbb{Z}/p\mathbb{Z})| \geq l$, it follows that there are l distinct lth roots of 1 in $(\mathbb{Z}/p\mathbb{Z})^*$. Since $H \simeq (\mathbb{Z}/p\mathbb{Z})^*$, we conclude that the surjective homomorphism $h \mapsto h^l\colon H \to H^l$ has kernel of order l, so $|H^l| = |H|/l$. Since $\mathcal{V}_3(\mathbb{Z}/p\mathbb{Z})$ and $\mathcal{V}_4(\mathbb{Z}/p\mathbb{Z})$ are nonempty, we find that $G_p \subseteq H^l$. Therefore

$$[H\colon G_p] \geq [H\colon H^l] = l.$$

So $l|G_p| \leq |H| = p - 1 < p$. ∎

LEMMA 8.6: *Let $f\colon \mathbb{N} \to \mathbb{Z}$ be a polynomial periodic function. Assume for all $N \in \mathbb{N}$, that $f(N) > 0$. Assume, for all $\epsilon > 0$, that there is some $N \in \mathbb{N}$ such that $f(N) < N\epsilon$. Then there exist integers $M, Q, R > 0$ such that, for all positive integers $N \in Q\mathbb{Z} + R$, we have $f(N) = M$.*

Proof: Choose an integer $Q > 0$ and polynomial functions $P_1, \ldots, P_Q\colon \mathbb{R} \to \mathbb{R}$ such that, for all integers $k \geq 0$, for all $r \in \{1, \ldots, Q\}$, we have $f(kQ+r) = P_r(k)$.

It suffices to show that one of the P_rs is constant, so assume, for all $r \in \{1, \ldots, Q\}$, that P_r is nonconstant. We wish to obtain a contradiction.

Fix some $r \in \{1, \ldots, Q\}$. Since the polynomial P_r is nonconstant, and since, for all integers $k \geq 0$, we have $P_r(k) = f(kQ + r) > 0$, it follows that there is some $\epsilon_r > 0$ such that, for all sufficiently large $x \geq 0$, we have $P_r(x) \geq \epsilon_r x$. It follows, for all sufficiently large $k \geq 0$, that

$$\frac{f(kQ+r)}{kQ+r} \geq \frac{P_r(k)}{k} \geq \epsilon_r.$$

By replacing ϵ_r by a smaller positive number, we may assume, for *all* $k \geq 0$, that

$$\frac{f(kQ+r)}{kQ+r} \geq \epsilon_r.$$

Let $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_Q\}$. Then, for all $N \in \mathbb{N}$, we have $f(N) \geq N\epsilon$, contradicting an assumption of Lemma 8.6. ∎

PROPOSITION 8.7: *The function $N \mapsto |G_N|$ is not polynomial periodic.*

Proof: Assume that $N \mapsto |G_N|$ is polynomial periodic; we wish to obtain a contradiction.

By Lemma 8.5 and Lemma 8.6, we see that there are integers $M, Q, R > 0$ such that, for all positive integers $N \in Q\mathbb{Z} + R$, we have $|G_N| = M$.

Let g_0 be as in Lemma 8.3. For all integers $N > 0$, $\mathrm{Rd}_N(g_0) \in G_N$, so $\mathrm{Rd}_N(g_0^M) = 1_{G_N}$. Therefore, for any positive integer $N \in Q\mathbb{Z} + R$, we have $g_0^M \in \ker(\mathrm{Rd}_N) = \mathcal{G}_0(N)$.

If $\mathcal{A}$ is any infinite subset of the set of positive integers, then

$$\bigcap_{N \in \mathcal{A}} \mathcal{G}_0(N) = \{I\},$$

so, using $\mathcal{A} := (Q\mathbb{Z} + R) \cap (0, \infty)$, we find that $g_0^M = I$. However, by Lemma 8.3, g_0 has infinite order, giving the contradiction. ∎

Now let M be a connected topological space with the homotopy type of a finite CW complex, let $\Gamma := \pi_1(M)$ and assume that $\Phi\colon \Gamma \to \mathcal{G}_0(\mathbb{Z})$ is a surjective homomorphism. For all integers $N > 0$, let $\Phi_N := \mathrm{Rd}_N \circ \Phi$.

Let $\tilde{M}$ be the universal cover of M. For all integers $N > 0$, let $M_N := \tilde{M} \times_{\Phi_N} G_N$ be the covering corresponding to $\ker(\Phi_N)$.

Let χ denote the Euler characteristic of M. For all integers $N > 0$, let χ_N denote the Euler characteristic of M_N; then $\chi_N = |G_N|\chi$. Then, by Proposition 8.7, we conclude that $N \mapsto \chi_N$ is not polynomial periodic.

For all integers $N > 0$ and all integers $q \geq 0$, let β_N^q denote the qth Betti number of M_N. Let $m := \dim(M)$. Since, for all integers $N > 0$, we have $\beta_N^0 = 1$, we conclude, for all integers $N > 0$, that

$$\chi_N = 1 + \sum_{q=1}^{m} (-1)^q \beta_N^q.$$

It follows, for some integer $q > 0$, that $N \mapsto \beta_N^q$ is not polynomial periodic, showing that the conclusion of Theorem 6.7 can fail if the algebraic group used to generate the tower is a torus. (Note that, because $\mathcal{G}_0(\mathbb{Z})$ is Abelian, it follows that every irreducible representation of $\mathcal{G}_0(\mathbb{Z})$ is one-dimensional; therefore, following the notation of Theorem 6.7, for all $N, q \in \mathbb{N}$, we have $\beta_N^{q;1} = \beta_N^q$.)

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